Green Functions for Nonlinear Klein-Gordon Kink Perturbation Theory

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We define a Green function for nonlinear Klein-Gordon theories in terms of the solutions to the eigenvalue equation obtained by linearizing the nonlinear wave equation about a static kink waveform. Analytic forms in terms of "modified" Lommel functions of two variables are derived for the sine-Gordon, phi-4 and double quadratic potentials. Asymptotic forms for the Green functions are obtained by investigating the asymptotic behavior of the modified Lommel functions. Methods for calculating the Lommel functions are also outlined.

I. INTRODUCTION

The influence of perturbations on the dynamics of kink solutions of nonlinear Klein-Gordon (NKG) models in 1+1 dimensions is a subject of particular importance in condensed matter contexts[1] where many different types of internal and/or external agents are responsible for spoiling the otherwise perfect (and boring!) propagation of kinks through the system of interest. Examples include impurities or other physical imperfections in the system[2], dissipative forces[2,3] and coupling to other degrees of freedom[3], external driving forces such as electric[4] or magnetic fields, stress fields[5], etc. In most cases of interest, the kinks involved carry some physically significant signature such as electric charge or spin, and hence can carry currents of various kinds which are important for the behavior of the system as a whole (e.g., conductivity[1,4]). Many of the physical systems of interest are modeled (sometimes justifiably) by nonlinear Klein-Gordon (NKG) Lagrangians such as the sine-Gordon(SG), ϕ^4, or double-quadratic (DQ) cases[1], among many others.

As a consequence of the importance of being able to determine the motion of kinks under perturbing influences such as those above, there have been several investigations over the last few years of either a general nature or having limited application to rather specific perturbations. One of the more useful approaches[3] has been to regard the kink of interest as an extended "particle" which obeys Newtonian dynamics at the classical level. Although there has been some controversy[6−9] regarding whether in fact the kink behaves as a Newtonian particle, this question has largely been resolved and one can adopt this Newtonian picture if care is taken to properly treat the behavior of the regions of the system far from the position of the kink.

The modern approach[10−12] based on these ideas is to regard the kink position as a collective coordinate and to perform a canonical transformation[12,13] to new coordinates, one of which is the kink "center-of-mass" position. The deviation of the full field from the pure kink profile is regarded as small (if the perturbing influence is small) and a systematic perturbation theory is employed in which successively higher powers of this deviation are included. The actual deviation of the kink position from its unperturbed value is not required to be small in this collective coordinate method, thus removing some secularities which occur in early versions[3] of this particle-like approach.

In the perturbation expansion method, it is convenient to employ a Green function technique[11,12] based on knowledge of the exact solutions for the small oscillations about the kink in an unperturbed system[1,3]. Until now this approach has been hindered by the lack of an analytic form for such Green functions since they involve integrals not found in the tables. In this paper we remedy this situation by reporting our closed-form evaluation of the Green functions for the three example systems mentioned above, namely SG, ϕ^4 and DQ. These forms involve modified Lommel functions of two variables[14] and since many of their properties have not to our knowledge been discussed in the literature, we examine some of the more useful of these, such as asymptotic expansions, in the present paper.

The remainder of the paper is organized as follows. Section II contains an introduction to the NKG models of interest, their kink solutions and the nature of small oscillations about the kinks in the pure system. The small oscillation solutions are then used in Section III to construct explicit, closed-form expressions for the Green functions of the three example systems in turn. In Section IV we discuss the asymptotic behavior of the Green functions by first investigating the asymptotic properties of the Lommel functions of two variables. Some of these results are new and are presented for the first time, to our knowledge, in this paper. In Section V we display and discuss some representative plots of the SG Green function as an example. Appendix A contains our evaluation of a generalized form of Hardy's integral for Lommel functions. Appendix B collects some of the properties of the modified Lommel functions of two variables while Appendix C describes some aspects of the numer-

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ical evaluation of modified Lommel functions and their asymptotic forms.

II. NONLINEAR KLEIN-GORDON KINKS AND THEIR SMALL OSCILLATIONS

In this section we briefly review the main features of solutions to the nonlinear Klein-Gordon class of field theories. The single-kink solutions to the wave equations along with small oscillations about these kinks will be described. The various quantities described in this section are collected in Table 1 for the sine-Gordon, ϕ^4, and double-quadratic potentials (this table corrects some errors in Table 1 of Ref. 15 and a similar error in Eq. (4.16b) in Ref. 1).

The general nonlinear Klein-Gordon Lagrangian we consider has the form

$$L = \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - V(\phi) \right\} , \tag{2.1}$$

where x and t are dimensionless space and time variables and $V(\phi)$ is the nonlinear potential. The nonlinear wave equation satisified by $\phi(x,t)$ is

$$\phi_{tt} - \phi_{xx} + V'(\phi) \ = 0, \tag{2.2}$$

where the prime on $V(\phi)$ denotes a derivative with respect to ϕ. Static single-kink solutions, $\phi_k(x)$, of Eq. (2.2) may be obtained by direct integration with the boundary conditions

$$\frac{d\phi_k(x)}{dx} \Big|_{x=\pm\infty} = 0 \ . \tag{2.3}$$

The static kink (+) and antikink (-) solutions are given by

$$x = \pm \frac{1}{\sqrt{2}} \int_{\phi_k(0)}^{\phi_k(x)} \frac{d\phi}{\sqrt{V(\phi)}} \ . \tag{2.4}$$

Moving solutions can be obtained by a Lorentz boost.

The equation governing the small oscillations about the static kink waveform is obtained by substituting

$$\phi(x,t) = \phi_k(x) + \psi(x,t) \ , \tag{2.5}$$

into Eq. (2.2) and linearizing in ψ:

$$\psi_{tt} - \psi_{xx} + V''[\phi_k(x)]\psi = 0 \ . \tag{2.6}$$

Here $V''[\phi_k(x)]$ denotes the second derivative of $V(\phi)$ with respect to ϕ evaluated for $\phi = \phi_k(x)$. Writing ψ as

$$\psi(x,t) = f(x)e^{-i\omega t} \ , \tag{2.7}$$

leads to the following eigenvalue equation:

$$-f_{xx} + V''[\phi_k(x)]f = \omega^2 f \ . \tag{2.8}$$

Due to the localized nature of the kink waveform $\phi_k(x)$, the function $V''[\phi_k(x)]$ varies mainly in the region of the kink center (assumed to be at $x = 0$) and approaches a constant (taken to be unity) far from the kink center:

$$V''[\phi_k(x)] \xrightarrow[|x|\to\infty]{} 1 \ . \tag{2.9}$$

Moreover, the function $V''[\phi_k(x)]$ has a minimum at x=0 such that

$$V''[\phi_k(0)] < 0 \ . \tag{2.10}$$

From these properties, we see that there exists a close analogy between Eq. (2.8) and the Schrödinger equation for a "particle" moving in a one-dimensional "potential well", $V''[\phi_k(x)]$. The "bound state(s)" and "continuum" states for this potential are of fundamental importance for statistical mechanics phenomenologies[15], perturbation theories for kink dynamics[2,15], and quantization procedures for kink states [10,12,13,17–19].

Since the Lagrangian (2.1) possesses translational invariance, the spectrum of the small oscillations about the single kink must contain a zero-frequency ($\omega = 0$) "translation" mode (Goldstone mode) which restores the translational invariance broken by the introduction of the kink. In addition to this translation mode there may be other discrete eigenvalues ("bound states") with frequencies between 0 and 1. These solutions correspond to "internal" oscillation modes in which the kink undergoes a harmonically varying shape change localized about the kink center. We denote these bound-state eigenfrequencies by $\omega_{b,1} \ldots \omega_{b,N}$ where N is the total number of bound states. The lowest of these is $\omega_{b,1} = 0$ since all other $\omega_{b,i}^2$ must be non-negative in order for the kink to be stable against small oscillations.

In addition to the bound states, there exist continuum states ("phonons") which are labelled by a wavevector k. These states have eigenvalues ω_k^2 given by

$$\omega_k^2 = 1 + k^2 \ , \tag{2.11}$$

which is precisely the dispersion relation for small oscillations in the absence of kinks.

The continuum states together with the bound-states form a complete set and satisify the completeness relation,

$$\sum_{i=1}^{N} f_{b,i}^*(x) f_{b,i}(x') + \int_{-\infty}^{\infty} dk f_k^*(x) f_k(x') = \delta(x - x') \ , \tag{2.12}$$

TABLE I: Various quantities for the ϕ^4, SG and DQ systems. $V(\phi)$ is the nonlinear potential, $\phi_k(x)$ is the kink (+) or anti-kink (-) solution, $V''[\phi_k(x)]$ is the potential which enters in the Schrödinger-like "phonon" equation [see Eq. (2.8)], and $f_{b,i}(x)$ and $f_k(x)$ are the bound and scattering states of $V''[\phi_k(x)]$ ($\omega_{b,1}=0$ for all three cases; $\omega_{b,2}=\sqrt{3}/2$ for the ϕ^4 potential).

	ϕ^4	SG	DQ
$V(\phi)$	$\frac{1}{8}\Big[\phi^2-1\Big]^2$	$1-\cos(\phi)$	$\frac{1}{2}\big[\lvert\phi\rvert-1\big]^2$
$\phi_k(x)$	$\tanh\Big[\pm\frac{x}{2}\Big]$	$4\tan^{-1}(e^{\pm x})$	$\pm sgn(x)\big[1-e^{-\lvert x\rvert}\big]$
$V''\big[\phi_k(x)\big]$	$1-\frac{3}{2}sech^2\big[\frac{x}{2}\big]$	$1-2sech^2(x)$	$1-2\delta(x)$
$f_{b,n}(x)$	$f_{b,1}(x)=\sqrt{\frac{3}{8}}sech^2\big[\frac{x}{2}\big]$ $f_{b,2}(x)=\sqrt{\frac{3}{4}}sech\big[\frac{x}{2}\big]tanh\big[\frac{x}{2}\big]$	$f_{b,1}(x)=\sqrt{\frac{1}{2}}sech(x)$	$f_{b,1}(x)=e^{-\lvert x\rvert}$
$f_k(x)$	$\frac{e^{ikx}\left[3tanh^2\left(\frac{x}{2}\right)-6iktanh\left(\frac{x}{2}\right)-(1+4k^2)\right]}{\sqrt{8\pi(1+k^2)(1+4k^2)}}$	$\frac{e^{ikx}\left[k+itanh(x)\right]}{\sqrt{2\pi(1+k^2)}}$	$\frac{\sin(kx)}{\sqrt{2\pi}}$ $\quad k<0$ $\frac{k\cos(kx)-sgn(x)\sin(kx)}{\sqrt{2\pi(1+k^2)}}$ $\quad k>0$

and the following orthogonality relations:

$$\int_{-\infty}^{\infty} dx f_{b,n}(x)f_{b,m}(x) \quad =\delta_{m,n} \ . \qquad (2.13a)$$

$$\int_{-\infty}^{\infty} dx f_k^*(x)f_{k'}(x) \ =\delta(k-k\prime) \ , \qquad (2.13b)$$

$$\int_{-\infty}^{\infty} dx f_k(x)f_{b,n}(x) \quad =0 \ . \qquad (2.13c)$$

Table 1 lists the nonlinear potentials, kink waveforms, small oscillation potentials, bound and scattering states for the SG, ϕ^4 and DQ potentials.

III. ANALYTIC EVALUATION OF THE GREEN FUNCTIONS

For the set$\{f_{b,i}(x), f_k(x)\}$ of solutions satisifying the "phonon" equation (2.8), we define the full Green function as:

$$\begin{aligned} &G(x,x',\tau) \\ &= \sum_{bound\ states} f_{b,i}^*(x)f_{b,i}(x')\int_{-\infty}^{\infty}\frac{d\omega\ e^{i\omega\tau}}{2\pi(\omega_{b,i}^2-\omega^2)} \\ &+\int_{-\infty}^{\infty} dk f_k^*(x)f_k(x')\int_{-\infty}^{\infty}\frac{d\omega\ e^{i\omega\tau}}{2\pi(\omega_k^2-\omega^2)} \ , \end{aligned} \qquad (3.1)$$

where $\tau\equiv t-t'$. Using the completeness relation (2.12), and the fact that the set $\{f_{b,i}(x), f_k(x)\}$ satisify equation (2.8), one can show that the full Green function satisfies the usual equation:

$$\big\{\partial_{tt}-\partial_{xx}+V''[\phi_k(x)]\big\}G(x,x',\tau)=\delta(x-x')\delta(\tau) \ . \qquad (3.2)$$

Once a set of boundary conditions is chosen the ω integral in Eq. (3.1) may be evaluated without choosing a particular set of $\{f_{b,i}(x), f_k(x)\}$. In this paper we choose retarded boundary conditions obtained by moving both of the poles in the ω integral above the real ω axis. Carrying out the ω integral yields:

$$G(x,x',\tau)=G_b(x,x',\tau)+G_p(x,x',\tau) \ , \qquad (3.3)$$

where $G_b(x,x',\tau)$ and $G_p(x,x',\tau)$ are the bound state and phonon contributions given by:

$$\begin{aligned} G_b(x,x',\tau) &= \theta(\tau)\Big\{\tau f_{b,1}^*(x)f_{b,1}(x') \\ &+ \sum_{i=2}^{N} f_{b,i}^*(x)f_{b,i}(x')\frac{\sin(\omega_i\tau)}{\omega_{b,i}}\Big\} \ , \qquad (3.4a) \\ G_p(x,x',\tau) &= \theta(\tau)\int_{-\infty}^{\infty} dk f_k^*(x)f_k(x')\frac{\sin(\omega_k\tau)}{\omega_k} \ , \qquad (3.4b) \end{aligned}$$

with N the number of bound states [if N=1, the second term is omitted from Eq. (3.4a)] and $\theta(\tau)$ is the Heavyside step function,

$$\theta(\tau)=\begin{cases}0, & -\infty<\tau<0 \\ 1, & 0\le\tau<\infty \ .\end{cases} \qquad (3.5)$$

In order to obtain explicit forms for these contributions to the Green function, one must insert the appropriate set of linearized solutions into Eqs. (3.4a) and (3.4b). As examples, we evaluate the phonon contribution for the SG, ϕ^4 and DQ potentials.

A. The SG Potential

Since the bound state contribution ($3.4a$) is already expressed in terms of known functions, we turn to the evaluation of the phonon contribution given in Eq. ($3.4b$). Inserting the functions $f_k(x)$ from the SG column of Table 1 into Eq. ($3.4b$) we have, after collecting common terms,

$$G_p^{SG}(x,x',\tau) = \theta(\tau)\{I_1 + \beta_2 I_2 + \beta_3 sgn(z) I_3\} \ , \quad (3.6)$$

where

$$I_1 = \frac{1}{\pi}\int_0^\infty \frac{dk}{\sqrt{1+k^2}}\cos(|z|k)\sin(\tau\sqrt{1+k^2}) \ , \quad (3.7a)$$

$$I_2 = \frac{1}{\pi}\int_0^\infty \frac{dk}{(1+k^2)^{\frac{3}{2}}}\cos(|z|k)\sin(\tau\sqrt{1+k^2}) \ , (3.7b)$$

$$I_3 = \frac{1}{\pi}\int_0^\infty \frac{dk}{(1+k^2)^{\frac{3}{2}}} k\sin(|z|k)\sin(\tau\sqrt{1+k^2}) \ , (3.7c)$$

with the definitions

$$\tau \equiv t - t' \quad , z \equiv x - x', \beta_2 \equiv \tanh(x)\tanh(x') - 1$$
$$\beta_3 \equiv \tanh(x') - \tanh(x) \ . \quad (3.8)$$

Since I_2 is uniformly convergent for all $|z|$ and τ, we may differentiate with respect to $|z|$ to obtain

$$I_3 = -\frac{dI_2}{d|z|} \ . \quad (3.9)$$

Therefore only $I_1$and I_2 need to be evaluated. These integrals may be evaluated by considering the integral $I(\mu)$ given by

$$I(\mu) = \frac{1}{\pi}\int_0^\infty \frac{dk}{\sqrt{\mu^2+k^2}}\cos(|z|k)\sin(\tau\sqrt{\mu^2+k^2}) \ , (3.10)$$
$$= \frac{\theta(\tau-|z|)}{2} J_0(\mu\sqrt{\tau^2-z^2}) \ , \quad (3.11)$$

where the integral is found in the tables[20]. The special case $I(1)$, is precisely the integral I_1. Since the derivative of the integrand of Eq. (3.10) is a continuous function of both μ and k, we may differentiate $I(\mu)$ with respect to μ to obtain

$$I_2 = \lim_{\mu\to 1}\Big\{-\frac{dI(\mu)}{d\mu} + \frac{\tau}{2\pi}\int_{-\infty}^\infty \frac{dk}{\mu^2+k^2} \times \cos(|z|k)\cos(\tau\sqrt{\mu^2+k^2})\Big\} \ , \quad (3.12)$$
$$= \frac{\theta(\tau-|z|)}{2}\sqrt{\tau^2-z^2}J_1(\sqrt{\tau^2-z^2}) + \frac{\tau}{2\pi}\int_{-\infty}^\infty \frac{dk}{1+k^2}\cos(|z|k)\cos(\tau\sqrt{1+k^2}) \ . (3.13)$$

In the integral remaining in Eq. (3.13) we substitute $k = \sinh(u)$, which gives us

$$\frac{\tau}{2\pi}\int_{-\infty}^\infty \frac{dk}{1+k^2}\cos(|z|k)\cos(\tau\sqrt{1+k^2})$$
$$= \frac{\tau}{2\pi}\int_{-\infty}^\infty \frac{du}{\cosh(u)}\cos[|z|\sinh(u)]\cos[\tau\cosh(u)] \ , \quad (3.14)$$
$$= \frac{\tau}{4\pi}\int_{-\infty}^\infty \frac{du}{\cosh(u)}\Big\{\cos\big[|z|\sinh(u)+\tau\cosh(u)\big] + \cos\big[\tau\cosh(u)-|z|\sinh(u)\big]\Big\} \ , \quad (3.15)$$
$$= \frac{\tau}{2\pi}\int_{-\infty}^\infty \frac{du\ e^u}{e^{2u}+1}\Big\{\cos\big[ae^u+be^{-u}\big]+\cos\big[ae^{-u}+be^u\big]\Big\} \ , \quad (3.16)$$
$$= \frac{\tau}{2\pi}\int_0^\infty \frac{dt}{t^2+1}\Big\{\cos\Big[at+\frac{b}{t}\Big]+\cos\Big[\frac{a}{t}+bt\Big]\Big\} \ , \quad (3.17)$$
$$= \frac{\tau}{\pi}\int_0^\infty \frac{dt}{t^2+1}\cos\big[at+\frac{b}{t}\big] \ , \quad (3.18)$$

where in passing from Eq. (3.17) to Eq. (3.18) we have let $t \to 1/t$ in the second cosine term and in Eq. (3.16) we have introduced the quantities

$$a \equiv \frac{\tau+|z|}{2} \ , \quad (3.19a)$$
$$b \equiv \frac{\tau-|z|}{2} \ . \quad (3.19b)$$

For $b < 0$ the integral in Eq. (3.18) is found in the tables[21] to be

$$\frac{1}{\pi}\int_0^\infty \frac{dt}{t^2+1}\cos\Big[at-\frac{|b|}{t}\Big] = \frac{1}{2}e^{-(a-b)}. \quad (3.20)$$

For $b > 0$, the integral in Eq. (3.18) may be expressed in terms of "modified" Lommel functions of two variables[14]. The "modified" functions, namely Lommel functions in which the first argument is pure imaginary, have not been found in the literature. Hence we introduce the notation $\Lambda_n(w,s)$ and $\Xi_n(w,s)$ for the modified Lommel functions and give their series representations in terms of Bessel functions:

$$\Lambda_n(w,s) \equiv i^{-n}U_n(iw,s) = \sum_{m=0}^\infty \Big(\frac{w}{s}\Big)^{2m+n} J_{2m+n}(s) \ , \quad (3.21a)$$

$$\Xi_n(w,s) \equiv i^{-n}V_n(iw,s) = \sum_{m=0}^\infty \Big(\frac{w}{s}\Big)^{-2m-n} J_{-2m-n}(s) \ . \quad (3.21b)$$

With these definitions, we write for $b > 0$

$$\frac{1}{\pi}\int_0^\infty \frac{dt}{t^2+1}\cos[at+\frac{|b|}{t}] = \frac{1}{2}e^{-(a-b)} - \Lambda_1(w,s)\ , \quad (3.22)$$

where

$$s \equiv \sqrt{\tau^2 - z^2}\ , \quad (3.23a)$$
$$w \equiv \tau - |z|\ . \quad (3.23b)$$

Combining Eqs. (3.20) and (3.22) we have for I_2 :

$$I_2 = \frac{1}{2}\tau e^{-|z|} + \theta(\tau - |z|)\Big\{\frac{sJ_1(s)}{2} - \tau\Lambda_1(w,s)\Big\}\ , \quad (3.24)$$

Using Eq. (B.6) from Appendix B we differentiate Eq. (3.24) with respect to $|z|$ which results in

$$\frac{dI_2}{d|z|} = -\frac{1}{2}\tau e^{-|z|} + \frac{\theta(\tau-|z|)}{2} \times\Big\{-(\tau+|z|)J_0(s) + 2\tau\Lambda_0(w,s)\Big\}\ . \quad (3.25)$$

In Eqs. (3.24) and (3.25), I_2 and its derivative appear to have terms which grow linearly in τ which is impossible in view of the integral representations in Eqs. (3.7). Using asymptotic expressions for the modified Lommel functions, we shall show in section IV that the large τ dependence is actually an inverse square root.

Writing the phonon contribution as

$$G_p^{SG}(x,x',\tau) = \theta(\tau)\Big\{I_1 + \beta_2 I_2 - \beta_3 sgn(z)\frac{dI_2}{d|z|}\Big\}\ , \quad (3.26)$$

we notice that with I_1, I_2 and $\frac{dI_2}{d|z|}$ given by Eqs. (3.11), (3.24) and (3.25), there is a term which does not vanish outside of the "light-cone" (i.e. a term which does not have $\theta(\tau - |z|)$ as a prefactor), namely

$$\theta(\tau)\frac{\tau e^{-|z|}}{2}\Big\{\beta_2 + sgn(z)\beta_3\Big\}\ . \quad (3.27)$$

One can show that this term may be rewritten as

$$-\theta(\tau)\tau f^*_{b,1}(x) f_{b,1}(x'). \quad (3.28)$$

Hence when the bound state contribution is added to Eq. (3.26) to obtain the full Green function, we are left with an expression which vanishes identically outside of the light-cone:

$$G^{SG}(x,x',\tau) = \frac{\theta(\tau-|z|)}{2}\Big\{J_0(s) + \beta_2\big[sJ_1(s) - 2\tau\Lambda_1(w,s)\big] -\beta_3 sgn(z)\big[-(\tau+|z|)J_0(s) + 2\tau\Lambda_0(w,s)\big]\Big\}\ , \quad (3.29)$$

explicitly demonstrating the retarded boundary conditions which have been applied.

B. The ϕ^4 Potential

With a slight generalization, the techniques used to evaluate the SG Green function may be applied to the ϕ^4 potential. Proceeding along the same lines, we write the phonon contribution as:

$$G_p^{\phi^4}(x,x',\tau) = \frac{\theta(\tau)}{4}\Big\{\gamma_0 I_0 - \gamma_1 sgn(z)\frac{dI_0}{d|z|} + \gamma_2 I_2 + \gamma_3 sgn(z)\frac{dI_2}{d|z|} + I_4\Big\}\ , \quad (3.30)$$

where I_2 and $\frac{dI_2}{d|z|}$ are given in Eqs. (3.24), (3.25) and

$$I_0 = \frac{1}{\pi}\int_0^\infty dk\frac{\cos(|z|k)\sin(\tau\sqrt{1+k^2})}{(1+k^2)^{\frac{3}{2}}(1+4k^2)}\ , \quad (3.31a)$$

$$I_4 = \frac{1}{\pi}\int_0^\infty dk\frac{(1+4k^2)\cos(|z|k)\sin(\tau\sqrt{1+k^2})}{(1+k^2)^{\frac{3}{2}}}\ , \quad (3.31b)$$

$$= 2\theta(\tau-|z|)J_0(s) - 3I_2\ , \quad (3.32)$$

$$\begin{aligned}
\gamma_0 &\equiv 9\big\{\tanh^2(y)\tanh^2(y') - \tanh(y)\tanh(y')\big\}\ ,\\
\gamma_1 &\equiv 18\big\{\tanh(y)\tanh^2(y') - \tanh^2(y)\tanh(y')\big\}\ ,\\
\gamma_2 &\equiv 9\tanh(y)\tanh(y') - 3\tanh^2(y) - 3\tanh^2(y')\ ,\\
\gamma_3 &\equiv 6\tanh(y) - 6\tanh(y')\ , \quad (3.33)\\
y &\equiv \frac{x}{2}\ ,\\
y' &\equiv \frac{x'}{2}\ ,
\end{aligned}$$

where Eq. (3.11) has been used to simplify Eq. $(3.31b)$. The remaining integral, I_0, may be reduced by partial fractions to

$$I_0 = \frac{4}{3\pi}\int_0^\infty dk\frac{\cos(|z|k)\sin(\tau\sqrt{1+k^2})}{\sqrt{1+k^2}\ (1+4k^2)} - \frac{I_2}{3}\ , \quad (3.34)$$

$$= \frac{4}{3}I_{01} - \frac{1}{3}I_2\ , \quad (3.35)$$

with I_{01} defined by

$$I_{01} = \frac{1}{\pi}\int_0^\infty dk\frac{\cos(|z|k)\sin(\tau\sqrt{1+k^2})}{\sqrt{1+k^2}\ (1+4k^2)}\ . \quad (3.36)$$

To evaluate I_{01} we again substitute $k = \sinh(u)$ which gives us

$$I_{01} = \frac{1}{\pi}\int_0^\infty du\frac{\cos[|z|\sinh(u)]\sin[\tau\cosh(u)]}{1+4\sinh^2(u)}\ , \quad (3.37)$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{tdt}{t^4-t^2+1}\sin\Big[at+\frac{b}{t}\Big]\ , \quad (3.38)$$

where in going from Eq. (3.37) to Eq. (3.38) substitutions similar to those made in Eqs. (3.14-18) have been made. Factoring the denominator of Eq. (3.38), we define

$$\beta_\pm^2 = -\, t_\pm^2 = -\beta_\mp = \frac{-1 \mp i\sqrt{3}}{2}\ , \qquad (3.39)$$

where $t_\pm^2$ are the roots of $t^4 - t^2 + 1$. Using partial fractions, we may write Eq. (3.38) as

$$\begin{aligned} I_{01} &= \frac{1}{2\pi i\sqrt{3}}\Big\{\int_0^\infty \frac{tdt}{t^2+\beta_+^2}\sin\Big[at+\frac{b}{t}\Big] \\ &\quad -\int_0^\infty \frac{tdt}{t^2+\beta_-^2}\sin\Big[at+\frac{b}{t}\Big]\Big\}\ , && (3.40)\\ &= \frac{-1}{2i\sqrt{3}}\big[J(\beta_-^2) - J^*(\beta_-^2)\big]\ , && (3.41)\\ &= \frac{-1}{\sqrt{3}} Im[J(\beta_-^2)]\ , && (3.42)\end{aligned}$$

where

$$J(\beta^2) \equiv -\ \frac{1}{\pi}\int_0^\infty \frac{tdt}{t^2+\beta^2}\sin\Big[at+\frac{b}{t}\Big]\ . \qquad (3.43)$$

The integral defined in Eq. (3.43) is a slight generalization of Hardy's integrals for Lommel functions[14,22]. The evaluation of $J(\beta^2)$ follows Hardy's with a few modifications and is presented in Appendix A for completeness. From Eq. (A.21) in Appendix A we have

$$J(\beta_-^2) = \frac{1}{2}e^{-(a\beta_- - \frac{b}{\beta_-})} - \theta(b)\Lambda_2\Big[\frac{2b}{\beta_-}, 2\sqrt{ab}\Big]\ , \qquad (3.44)$$

$$= \frac{1}{2}e^{-\frac{1}{2}(|z|+i\sqrt{3}\tau)} - \theta(\tau-|z|)\Lambda_2(\beta_+ w, s)\ . \qquad (3.45)$$

Therefore, we have for I_{01} :

$$I_{01} = \frac{1}{2\sqrt{3}}e^{-\frac{|z|}{2}}\sin(\omega_{b,2}\tau) + \frac{\theta(\tau-|z|)}{\sqrt{3}} Im[\Lambda_0(\beta_+ w, s)]\ , \qquad (3.46)$$

where

$$\omega_{b,2} \equiv \frac{\sqrt{3}}{2}\ , \qquad (3.47)$$

and we have used

$$\begin{aligned} &Im[\Lambda_2(\beta_+ w,s)] = \\ &\quad Im[\Lambda_0(\beta_+ w,s) + J_0(s)] = Im[\Lambda_0(\beta_+ w,s)]\ . \qquad (3.48)\end{aligned}$$

From Eq. (3.30) we see that we need a derivative of I_0, and hence I_{01}, with respect to $|z|$. Using Eq. (B.6) and Eqs. (B.16) from Appendix B we have

$$\begin{aligned} \frac{dI_{01}}{d|z|} &= \frac{-1}{4\sqrt{3}}e^{-\frac{|z|}{2}}\sin(\omega_{b,2}\tau) \\ &\quad - \frac{\theta(\tau-|z|)}{2\sqrt{3}} Im[\Lambda_1(\beta_+ w,s)]\ , \qquad (3.49)\end{aligned}$$

where

$$\frac{\beta_+^2+1}{\beta_+} = 1\ , \qquad (3.50)$$

has also been used. Collecting all of the pieces, we write for the phonon contribution:

$$\begin{aligned} &G_p^{\phi^4}(x,x',\tau) \\ &= \frac{\theta(\tau)}{4}\Big\{\frac{4}{3}\gamma_0 I_{01} - \frac{4}{3}\gamma_1 sgn(z)\frac{dI_{01}}{d|z|} \\ &\quad + \big[\gamma_2 - \frac{\gamma_0}{3} - 3\big] I_2 \\ &+\ sgn(z)\big[\frac{\gamma_1}{3} + \gamma_3\big]\frac{dI_2}{d|z|} \\ &\quad + 2\theta(\tau-|z|)J_0(s)\Big\}. \qquad (3.51)\end{aligned}$$

As in the sine-Gordon case one may show that when we combine the "non-retarded" pieces of the phonon contribution, we obtain exactly the negative of the bound state contribution; specifically we have

$$\begin{aligned} \frac{1}{8}\ &\big[\gamma_2 - \frac{\gamma_0}{3} - 3\big]\tau e^{-\frac{|z|}{2}} - \frac{sgn(z)}{8}\big[\frac{\gamma_1}{3}+\gamma_3\big]\tau e^{-\frac{|z|}{2}}\ , \\ &= -\tau f_{b,1}^*(x) f_{b,1}(x') \qquad (3.52a)\\ \frac{1}{6\sqrt{3}}\ &e^{-\frac{|z|}{2}}\sin(\omega_{b,2}\tau)\gamma_0 + \frac{1}{12\sqrt{3}}e^{-\frac{|z|}{2}}\sin(\omega_{b,2}\tau) sgn(z)\gamma_1 \\ &= -\frac{\sin(\omega_{b,2}\tau)}{\omega_{b,2}} f_{b,2}^*(x) f_{b,2}(x')\ . \qquad (3.52b)\end{aligned}$$

With the "non-retarded" portion cancelled by the bound state contribution, we have for the full Green function

$$\begin{aligned} &G^{\phi^4}(x,x',\tau) = \\ &\theta(\tau-|z|)\Big\{\frac{1}{3\sqrt{3}} Im\Big[\gamma_0\Lambda_0(\beta_+ w,s) \\ &+ \frac{1}{2}\gamma_1 sgn(z)\Lambda_1(\beta_+ w,s)\Big] \\ &+ \frac{1}{8}\big[\gamma_2 - \frac{\gamma_0}{3} - 3\big]\big[sJ_1(s) - 2\tau\Lambda_1(w,s)\big] \\ &+ \frac{sgn(z)}{8}\big[\frac{\gamma_1}{3}+\gamma_3\big]\big[-(\tau+|z|)J_0(s) \\ &7 + 2\tau\Lambda_0(w,s)\big] + \frac{1}{2}J_0(s)\Big\}. \qquad (3.53)\end{aligned}$$

C. The DQ Potential

As a final example, we evaluate the DQ Green function. The phonon contribution in this case is

$$G_p^{DQ}(x,x',\tau) = \theta(\tau-|z|)\Big\{I_1 - \Big[I_2(z_+) - \frac{dI_2(z_+)}{dz_+}\Big]\Big\}\ , \qquad (3.54)$$

where I_1 is given in Eq. (3.11) [with $\mu = 1$] and $I_2(z_+)$ is given in Eq. (3.24) with $|z|$ replaced by $z_+ \equiv |x| + |x'|$.

Factoring out the non-retarded piece we have

$$\begin{aligned} & G^{DQ}(x,x',\tau) \\ &= \frac{\theta(\tau-|z|)}{2}\Big\{J_0(s)-s_+J_1(s_+) \\ &\quad +2\tau\Lambda_1(w_+,s_+) \\ &\quad +(\tau+z_+)J_0(s_+)+2\tau\Lambda_0(w_+,s_+)\Big\}\ , \end{aligned} \tag{3.55}$$

with

$$z_+\equiv|x|+|x'| \quad , \quad w_+\equiv\tau-z_+ \quad , \quad s_+\equiv\sqrt{\tau^2-z_+^2}\ . \tag{3.56}$$

All three of the Green functions derived above have been checked against numerical integration. Over a large range of values for x,x' and τ, we find agreement to 8 significant digits, which is presently the accuracy of our routines which compute the modified Lommel functions. In addition we have applied the small oscillation operator [see Eq. (3.2)] to each of the analytic expressions which, after some tedious algebra, yield the appropriate delta functions. To obtain a final check, we note that by using the orthogonality relation in Eq. (2.13c) and linear superposition, we see that the phonon contribution to the Green functions must be orthogonal to the bound state(s). Numerical integrations confirm this property for all three Green functions.

IV. IV ASYMPTOTIC BEHAVIOR

To obtain asymptotic expressions ($\tau\to\infty$) for the Green functions, we must first find the appropriate limits of the modified Lommel functions. In Appendix C we examine $\Lambda_0(w,s)$ and $\Lambda_1(w,s)$ in the limit as $s\to\infty$ while $w/s\to1$, which, when w and s are related to τ and z by Eqs. (3.23), corresponds to $\tau\gg|z|$. This limit is interesting because the expressions for the phonon contributions to the Green functions have a term linear in τ which, in view of the integral expressions, must be cancelled by the other terms.

Since all of the Green functions are expressible in terms of the integrals I_{01},I_2 and their derivatives with respect to $|z|$, we consider the asymptotic expressions for these quantities first and then combine them to obtain the limits for the Green functions.

To apply the results of Appendix C we must first recast these results in terms of the variables τ and z which are related to w and s by

$$w=\beta(\tau-|z|) \quad , \quad s=\sqrt{\tau^2-z^2}\ , \tag{4.1}$$

where β is either unity or β_+. From Eqs. (C.31) and (C.32) of Appendix C, we have for $\beta=1$,

$$\Lambda_0(w,s)\approx\frac{J_0(s)}{2}+\frac{e^{-|z|}}{2}+\frac{|z|}{2\tau}\sqrt{\frac{2}{\pi s}}\left\{\cos(s-\frac{\pi}{4})\Big[1+\frac{2R_4(1,\kappa)}{(8s)^2}\Big]+\sin(s-\frac{\pi}{4})\frac{2R_2(1,\kappa)}{8s}\right\}+O(\tau^{-\frac{7}{2}})\ , \tag{4.2}$$

$$\begin{aligned}\Lambda_1(w,s)\approx\frac{e^{-|z|}}{2}-\frac{s}{2\tau}\sqrt{\frac{2}{\pi s}}\Big\{ & \cos(s-\frac{\pi}{4})\Big[\frac{2[R_2(1,\kappa)-2]}{8s}-\frac{40R_4(1,\kappa)}{(8s)^3}\Big] \\ - & \sin(s-\frac{\pi}{4})\Big[1+\frac{2[R_4(1,\kappa)+12R_2(1,\kappa)]}{(8s)^2}\Big]\Big\}+O(\tau^{-\frac{9}{2}})\ ,\end{aligned} \tag{4.3}$$

where $\kappa\equiv w/s$, R_2 and R_4 are defined in Eqs. (C.29), (C.30), and we have used [see Eqs. (C.13)]

$$\epsilon(1,\kappa)=\frac{|z|}{s}\ ,\sigma_1(1,\kappa)=\frac{\tau}{2s}\ ,\sigma_2(1,\kappa)=\frac{\tau}{2|z|}\ ,\ \frac{\sigma_1(1,\kappa)}{\sqrt{1+\epsilon^2(1,\kappa)}}=\frac{1}{2}\ ,\ \frac{\epsilon(1,\kappa)\sigma_2(1,\kappa)}{1+\epsilon^2(1,\kappa)}=\frac{s}{2\tau}\ . \tag{4.4}$$

Inserting the expression for $\Lambda_1(w,s)$ given by Eq. (4.3) into Eq. (3.24), we see that the linear τ dependence exactly cancels (for large τ and $\tau\gg|z|$, both $\theta(\tau-|z|)$ and $\theta(\tau)$ are unity), leaving us with:

$$\begin{aligned}I_2\approx\frac{sJ_1(s)}{2}+\frac{s}{2}\sqrt{\frac{2}{\pi s}}\Big\{ & \cos(s-\frac{\pi}{4})\Big[\frac{2[R_2(1,\kappa)-2]}{8s}-\frac{40R_4(1,\kappa)}{(8s)^3}\Big] \\ - & \sin(s-\frac{\pi}{4})\Big[1+\frac{2[R_4(1,\kappa)+12R_2(1,\kappa)]}{(8s)^2}\Big]\Big\}+O(\tau^{-\frac{7}{2}})\ .\end{aligned} \tag{4.5}$$

In Eq. (4.5), I_2 now seems to have a $\sqrt{s}$ and therefore $\sqrt{\tau}$ dependence, however this again exactly cancels when $J_1(s)$

is expanded in its asymptotic series resulting in:

$$I_2 \approx \frac{1}{2}\sqrt{\frac{2}{\pi s}}\Big\{ \quad \sin(s-\frac{\pi}{4})\Big[\frac{15-4[R_4(1,\kappa)+12R_2(1,\kappa)]}{16(8s)}\Big] + \quad \cos(s-\frac{\pi}{4})\Big[\frac{2R_2(1,\kappa)-1}{8}+\frac{5[21/16-R_4(1,\kappa)]}{(8s)^2}\Big]\Big\} + O(\tau^{-\frac{7}{2}}) \ . \tag{4.6}$$

Similarly we have

$$\frac{dI_2}{d|z|} \approx \frac{|z|}{2}\sqrt{\frac{2}{\pi s}}\Big\{\cos(s-\frac{\pi}{4})\Big[\frac{9+4R_2(1,\kappa)}{2(8s)^2}\Big]+\sin(s-\frac{\pi}{4})\Big[\frac{2R_2(1,\kappa)-1}{(8s)}\Big] \ \Big\} + O(\tau^{-\frac{7}{2}}) \ . \tag{4.7}$$

Next we turn to the I_{01} expression which involves modified Lommel functions evaluated at $\beta_+ w$ and s. With $\beta = \beta_+$, $\epsilon(\beta,\kappa)$, $\sigma_1(\beta,\kappa)$ and $\sigma_2(\beta,\kappa)$ become

$$\epsilon(\beta_+,\kappa) = \frac{|z|+i\sqrt{3}\tau}{2s} \ ,\sigma_1(\beta_+,\kappa) = \frac{\tau+i\sqrt{3}\ |z|}{4\pi s} \ ,\sigma_2(\beta_+,\kappa) = \frac{\kappa}{2s}\frac{(\tau+i\sqrt{3}|z|)(\tau+|z|)}{|z|+i\sqrt{3}\tau} \ . \tag{4.8}$$

Inserting Eqs. (4.8) into Eqs. (C.31) and (C.32), we have

$$\Lambda_0(\beta_+ w,s) \approx \frac{1}{2}e^{-\frac{|z|}{2}}e^{i\omega_{b,2}t}+\frac{1}{2}\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}}\sqrt{\frac{2}{\pi s}}\Big\{ \quad \cos(s-\frac{\pi}{4})\Big[1+\frac{2R_4(\beta_+,\kappa)}{(8s)^2}\Big] + \quad \sin(s-\frac{\pi}{4})\frac{2R_2(\beta_+,\kappa)}{(8s)}\Big\} + O(\tau^{-\frac{7}{2}}) \ , \tag{4.9}$$

$$\Lambda_1(\beta_+ w,s) \approx \frac{1}{2}e^{-\frac{|z|}{2}}e^{-i\omega_{b,2}t}-\frac{1}{2}\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}}\sqrt{\frac{2}{\pi s}} \quad \Big\{\cos(s-\frac{\pi}{4})\Big[\frac{2[R_2(\beta_+,\kappa)-2]}{8s}-40\frac{R_4(\beta_+,\kappa)}{(8s)^3}\Big] -\sin(s-\frac{\pi}{4})\Big[1+\frac{2[R_4(\beta_+,\kappa)+12R_2(\beta_+,\kappa)]}{(8s)^2}\Big]\Big\} + O(\tau^{-\frac{9}{2}}) \ , \tag{4.10}$$

where we have used

$$\frac{\sigma_1(\beta_+,\kappa)}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}} = \frac{1}{2} \qquad , \qquad \frac{\epsilon(\beta_+,\kappa)\sigma_2(\beta_+,\kappa)}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}} = \frac{1}{2} \ . \tag{4.11}$$

When Eq.(4.9) is inserted into the expression for I_{01}, the oscillatory term in τ cancels leaving

$$I_{01} \approx \frac{1}{2\sqrt{3}}Im\Big\{\frac{1}{\sqrt{1+\epsilon^2(1,\kappa)}}\sqrt{\frac{2}{\pi s}}\Big[\quad \cos(s-\frac{\pi}{4})(1+\frac{2R_4(\beta_+,\kappa))}{(8s)^2}) + \quad \sin(s-\frac{\pi}{4})\frac{2R_2(\beta_+,\kappa)}{(8s)}\Big]\Big\} + O(\tau^{-\frac{7}{2}}) \ , \tag{4.12}$$

and

$$\frac{dI_{01}}{d|z|} \approx \frac{1}{2\sqrt{3}}Im\Big\{\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}}\sqrt{\frac{2}{\pi s}}\Big[\quad \cos(s-\frac{\pi}{4})\Big(\frac{2[R_2(\beta_+,\kappa)-2]}{8s}\Big) - \quad \sin(s-\frac{\pi}{4})\Big(1+\frac{2[R_4(\beta_+,\kappa)+12R_2(\beta_+,\kappa)]}{(8s)^2}\Big)\Big]\Big\} + O(\tau^{-\frac{7}{2}}) \ . \tag{4.13}$$

Now all of the contributions are at hand to obtain, through $O(\tau^{-\frac{7}{2}})$, the asymptotic forms for the Green functions. However, since the expressions are lengthy and not particularly illuminating, we list only the leading terms. Due to the simple analytic form of the bound state contribution, we list only the phonon portions:

$$G_p^{SG}(x,x',\tau) \approx \sqrt{\frac{2}{\pi s}}\left\{\cos(s-\frac{\pi}{4})+\frac{1}{8s}\sin(s-\frac{\pi}{4})\right\}+O(\tau^{-\frac{5}{2}}) , \tag{4.14}$$

$$\begin{aligned} G_p^{\phi^4}(x,x',\tau) \approx \sqrt{\frac{2}{\pi s}}\Bigg\{&\cos(s-\frac{\pi}{4})\Big[\frac{\gamma_0}{6\sqrt{3}}Im\Big(\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}}\Big)+\frac{1}{8}\Big(\gamma_2-\frac{\gamma_0}{3}-3\Big)\Big(\frac{2R_2(1,\kappa)-1}{8}\Big)+2\Big] \\ &-\sin(s-\frac{\pi}{4})\Big[\frac{\gamma_1 sgn(z)}{12\sqrt{3}}Im(\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}})\Big]\Bigg\}+O(\tau^{-\frac{3}{2}}) , \end{aligned} \tag{4.15}$$

$$G_p^{DQ}(x,x',\tau) \approx \sqrt{\frac{2}{\pi s}}\cos(s-\frac{\pi}{4})-\frac{1}{2}\sqrt{\frac{2}{\pi s_+}}\cos(s_+-\frac{\pi}{4})\Big[\frac{2R_2(1,\kappa_+)-1}{8}\Big]+O(\tau^{-\frac{3}{2}}) , \tag{4.16}$$

where in Eq. (4.16), $\kappa_+ \equiv w_+/s_+$.

One may notice that although we have shown that there is no linear τ term in the phonon contributions to the Green functions, the full Green functions have a linear τ term due to the first bound state, namely,

$$\theta(\tau)\tau f_{b,1}^*(x) f_{b,1}(x') . \tag{4.17}$$

This term may be understood by realizing that when computing the response of a soliton to a perturbation, the effect of this term is to produce a coefficient of the translation mode $f_{b,1}(x)$ which increases with time. Therefore, the soliton will move from its initial position as time progresses. Hence in this case, the linear term is associated with the translation of the soliton.

The secularity referred to in the introduction is made evident by the linear τ behavior in the coefficient of the translation-mode contribution to the full Green function. Indeed, the use of the full Green function in a perturbation theory of kink dynamics in the presence of external influences is equivalent to the procedure introduced by Fogel et al.[2]. The use of the collective-coordinate method[10−13] avoids the secularity associated with the translation mode since only the "phonon" part of the Green function is employed[10−13] [together with the contribution from other bound states, if any ($N \geq 2$)].

V. REPRESENTATIVE PLOTS

To illustrate the behavior of the Green functions, we present several plots of the *phonon* part of the SG Green function [plots for the other Green functions derived look very similar] in the $x-x'$ plane for different values of τ. The numerical values for these plots are easily obtained from the formulae in Appendix C.

Since the Green functions depend only on $\tau = t-t'$, we are free to choose t' and let t be fixed by τ and t'. Choosing $t'=0$, Figure 1 shows the evolution for the phonon contribution as time progresses. To interpret these plots, recall that $G(x,x',t,t')$ may be viewed as the response of the system at (x,t) due to a delta function source at (x',t'). Fixing $x'=8$ in Figure 1a, we move in the direction of increasing x, starting at $x=0$. Until x is on the order of 2, $G(x,x',\tau)$ is zero, meaning that the disturbance has not yet had enough time to propagate from $x=8$ to $x<2$ (or $x>14$). For $\tau=4$, time has progressed (recall we have fixed $t'=0$) and the disturbance has propagated out further. At $t=8$ the pulse reaches $x=8$. In Figures 1.e to 1.h the pulse has propagated off the scales, leaving behind "ripples". As τ further increases the amplitude continues to decrease in accord with the asymptotic behavior derived in section IV.

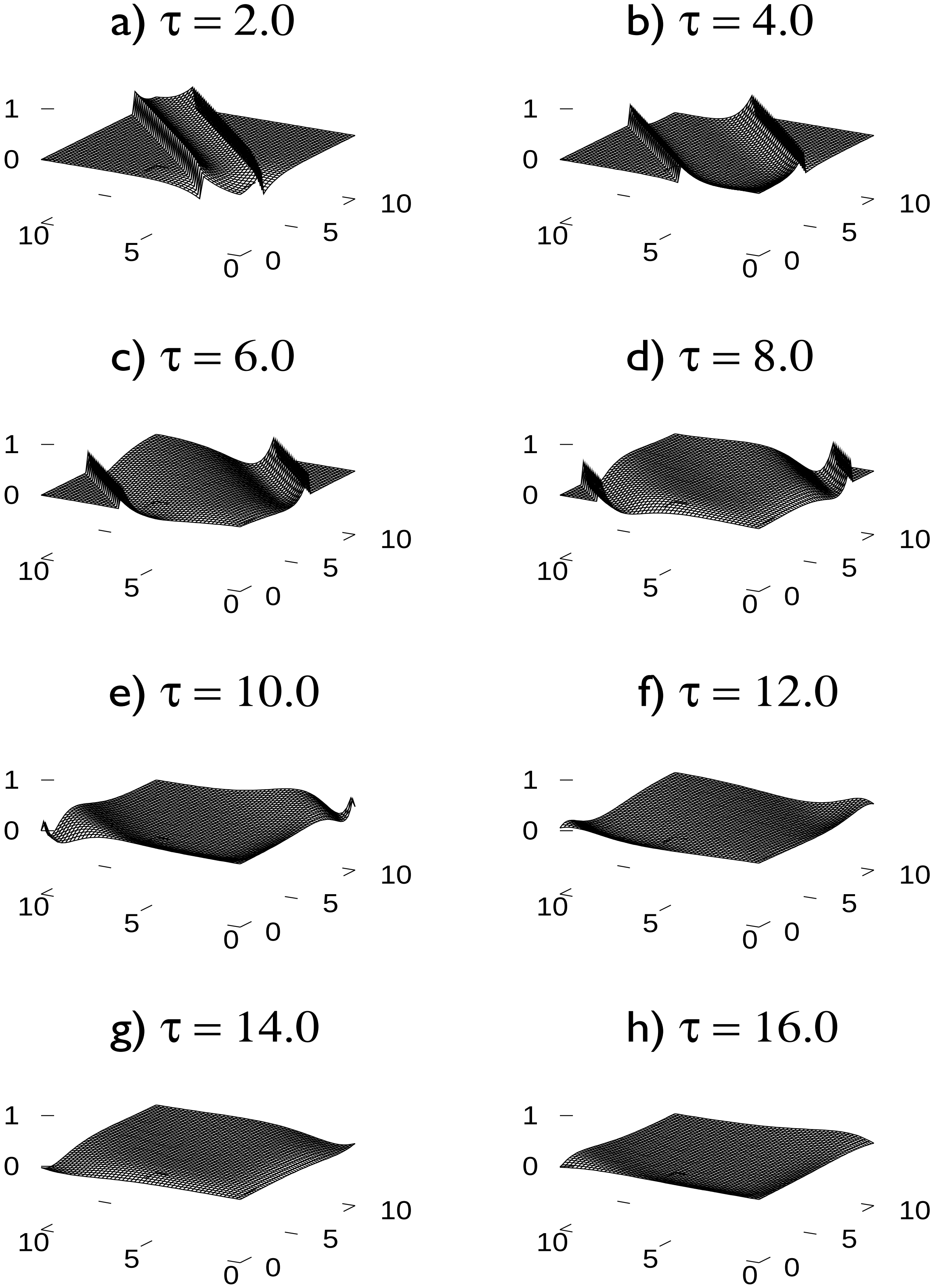
If one were to follow the procedure outlined in the preceding paragraph with $x'=3$, one would note that before the pulse arrives at a particular position, the Green function is not zero. This is because we have plotted the phonon contribution, which has a non-retarded part which exactly cancels the bound state contribution. It is this non-retarded part which gives a non-zero value for the phonon contribution to the Green function "before the pulse arrives". We see this only near $x=x'=0$ because the bound state contribution is proportional to $e^{-|z|}$ [SG], $sech(x)sech(x')$ [ϕ^4] or $e^{-|x|}e^{-|x'|}$ [DQ].

[*Note added in proof:*]Recently we have been able to obtain analytic expressions for the Laplace transform of the product of the Lommel function $\Lambda_n(w,s)$ and the step function $\theta(\tau-|z|)$, with w and s related to τ and z by Eqs. (3.23). Specifically we have

$$\int_0^\infty d\tau e^{-\bar{s}\tau}\theta(\tau-|z|)\Lambda_n(w,s) = \frac{1}{2s}\frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{\sqrt{\bar{s}^2+1}}\left[\frac{1}{\sqrt{\bar{s}^2+1}+\bar{s}}\right]^{n-1} , \tag{5.1}$$

where $\bar{s}$ is the Laplace transform variable. This leads to the remarkably simple expression for the Laplace trans-

FIG. 1: The time evolution of the phonon contribution to the SG Green function $G(x, x', t - t')$ in the x-x' plane. Here we have chosen $t' = 0$, therefore $\tau = t$. In figures $a \to d$ we see a disturbance "propagating outward". Figures a and b show the non-retarded portion near $x = x' = 0$. In figures $e \to h$ the pulse has moved off of our scales, leaving behind undulations which decrease with increasing time.



form of the sine-Gordon Green's function

$$G^{SG}(x,x';\bar{s}) \equiv \int_0^\infty d\tau G^{SG}(x,x',\tau)e^{-\bar{s}\tau}$$
$$= \frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{2}\left\{\frac{1}{\sqrt{\bar{s}^2+1}} - \frac{\beta_2}{\bar{s}^2\sqrt{\bar{s}^2+1}} - \frac{\beta_3 sgn(z)}{\bar{s}^2}\right\}, \tag{5.2}$$

with similar expressions for the ϕ^4 and DQ Green's functions

APPENDIX A: EVALUATION OF THE INTEGRAL $J(\beta^2)$

The integral $J(\beta^2)$ [Eq. (A.1)] differs from Hardy's integral for Lommel functions[14,22] only in that in the denominator, the t^2+1 is replaced by $t^2+\beta^2$. The only restriction placed on β is that $Re(\beta)>0$. We first consider the case in which $b<0$ for which we have from the tables[23],

$$J(\beta^2) = \frac{1}{\pi}\int_0^\infty \frac{tdt}{t^2+\beta^2}\sin\Big[at+\frac{b}{t}\Big] = \frac{1}{2}e^{-(a\beta-\frac{b}{\beta})}\ , \tag{A1}$$

where the restriction $Re(\beta)>0$ is required.

For $b>0$ we distinguish between $b<a$ and $b>a$. The latter may be reduced to the $b<a$ case by using the relation[24],

$$\frac{1}{\pi}\int_0^\infty \frac{tdt}{t^2+\beta^2}\sin\Big[at+\frac{b}{t}\Big]$$
$$= J_0(2\sqrt{ab}) - \frac{1}{\pi}\int_0^\infty \frac{tdt}{t^2+\frac{1}{\beta^2}}\sin\Big[\frac{a}{t}+bt\Big]\ . \tag{A2}$$

Therefore we need only consider $b<a$. We may further restrict ourselves to $|\beta|=1$ by writing $\beta=|\beta|e^{i\varphi}$ which allows us to write

$$J(\beta^2) = \frac{1}{\pi}\int_0^\infty \frac{tdt}{|\beta|^2\big[\frac{t^2}{|\beta|^2}+e^{2i\varphi}\big]}\sin\Big[at+\frac{b}{t}\Big]\ , \tag{A3}$$
$$= \frac{1}{\pi}\int_0^\infty \frac{tdt}{t^2+e^{2i\varphi}}\sin\Big[a't+\frac{b'}{t}\Big], \tag{A4}$$

where a' and b' are a and b scaled by $1/|\beta|$. Therefore, with $b<a$ and $|\beta|=1$, we define

$$x \equiv 2\sqrt{ab} \quad , \quad c \equiv \frac{1}{\beta}\sqrt{\frac{b}{a}}\ , \tag{A5}$$

in terms of which we may write $J(\beta^2)$ as

$$J(\beta^2) = \frac{1}{\pi}\int_0^\infty \frac{tdt}{t^2+\beta^2}\sin\Big[\frac{x}{2}\Big(t\sqrt{\frac{a}{b}}+\frac{1}{t}\sqrt{\frac{b}{a}}\Big)\Big]\ , \tag{A6}$$
$$= \frac{c}{\pi}\int_{-\infty}^\infty \frac{e^u du}{ce^u+\frac{1}{ce^u}}\sin\big[x\cosh(u)\big]\ , \tag{A7}$$
$$= \frac{c}{\pi}\int_0^\infty du\Big\{\frac{e^{-u}}{ce^{-u}+(ce^{-u})^{-1}} + \frac{e^u}{ce^u+(ce^u)^{-1}}\Big\}\sin\big[x\cosh(u)\big]\ , \tag{A8}$$
$$= \frac{1}{2\pi}\int_1^\infty \frac{d\tau}{\sqrt{\tau^2-1}}\frac{c^2-1+2\tau^2}{\theta^2+\tau^2}\sin(x\tau)\ , \tag{A9}$$

with

$$\theta \equiv \frac{1}{2}(c-\frac{1}{c}) = \frac{c'^2+1}{2c'}\Big\{\frac{c'^2-1}{c'^2+1}Re(\beta) - iIm(\beta)\Big\}\ , \tag{A10}$$

$$c' \equiv \sqrt{\frac{b}{a}}\ . \tag{A11}$$

Since $Re(\beta)>0$ and $c'<1$, θ is never pure imaginary; therefore θ^2 does not lie on the negative real axis and the only poles of the integrand in Eq. (A.9) are at $\tau=\pm1$. We evaluate Eq. (A.9) by considering the contour integral $\Gamma(\beta^2)$ given by

$$\Gamma(\beta^2) \equiv \int_\Gamma \frac{dz\ e^{ixz}}{\sqrt{z^2-1}}\frac{c^2-1+2z^2}{\theta^2+z^2}\ . \tag{A12}$$

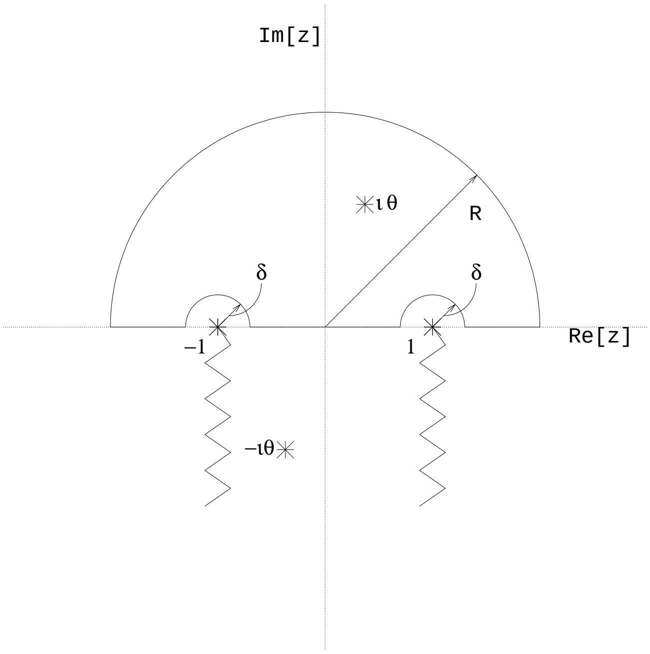
With the branch cuts chosen as in Figure 2, $\Gamma(\beta^2)$ becomes

$$\Gamma(\beta^2) = 2i\int_1^\infty \frac{d\tau\ sin(x\tau)}{\sqrt{\tau^2-1}}\frac{c^2-1+2\tau^2}{\theta^2+\tau^2}$$
$$-\ 2i\int_{-1}^1 \frac{d\tau\ e^{ix\tau}}{\sqrt{1-\tau^2}}\frac{c^2-1+2\tau^2}{\theta^2+\tau^2}\ . \tag{A13}$$

Therefore we have for $J(\beta^2)$,

$$J(\beta^2) = \frac{1}{2\pi i}\frac{\Gamma(\beta^2)}{2}$$
$$+\ \frac{1}{2\pi}\int_0^1 \frac{d\tau\ \cos(x\tau)}{\sqrt{1-\tau^2}}\frac{c^2-1+2\tau^2}{\theta^2+\tau^2}\ , \tag{A14}$$
$$= \frac{Res[f(z);-i\theta]}{2}$$
$$+\frac{1}{2\pi}\int_0^{\frac{\pi}{2}} d\varphi\ \cos[x\cos(\varphi)]\frac{c^2-1+2\cos^2(\varphi)}{\theta^2+\cos^2(\varphi)}, \tag{A15}$$

FIG. 2: The contour Γ for the evaluation of the integral $J(\beta^2)$.



where $Res[f(z);-i\theta]$ is the residue of $f(z)$ evaluated at $-i\theta$ with $f(z)$ given by the integrand of Eq. (A.12). In writing Eq. (A.14) we have used the fact the contributions to $\Gamma(\beta^2)$ from the large and small semicircles vanish when $R\to\infty$ and $\delta\to 0$ respectively. Evaluating the residue at the simple pole $-i\theta$ we have

$$Res[f(z);-i\theta] = e^{-(a\beta-\frac{b}{\beta})} \ . \tag{A16}$$

The remaining integral in Eq. (A.15) may be evaluated by noting that

$$\frac{c^2-1+2\cos^2(\varphi)}{\theta^2+\cos^2(\varphi)} = -4\sum_{k=1}^{\infty}(ic)^{2k}\cos(2k\varphi) \ . \tag{A17}$$

Since $c<1$, the sum in Eq. (A.17) is uniformly convergent and we may insert it into Eq. (A.15) and integrate term by term. We also make the substitution

$$\cos[x\cos(\varphi)] = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)\cos(2n\varphi) \ , \tag{A18}$$

The double sum resulting from substitution of Eqs. (A.17) and (A.18) into Eq. (A.15) is reduced to a single sum by orthogonality of the functions $\{\cos(2n\varphi)\}$ on $[0,\frac{\pi}{2}]$, leaving

$$\begin{aligned}\frac{1}{2\pi}\int_0^{\infty}\frac{d\tau\cos(x\tau)}{\sqrt{1-\tau^2}}\frac{c^2-1+2\tau^2}{\theta^2+\tau^2} \\ &= -\sum_{k=1}^{\infty}c^{2k}J_{2k}(x) \ , &\text{(A19)}\\ &= -\Lambda_2\Big[\frac{2b}{\beta},2\sqrt{ab}\Big] \ . &\text{(A20)}\end{aligned}$$

Finally collecting Eqs. (A.1), (A.16) and (A.20) we have

$$J(\beta^2) = \frac{1}{2}e^{-(a\beta-\frac{b}{\beta})} - \theta(b)\Lambda_2\Big[\frac{2b}{\beta},\sqrt{2ab}\Big] \ . \tag{A21}$$

APPENDIX B: APPENDIX B. PROPERTIES OF LOMMEL FUNCTIONS OF TWO VARIABLES

In this appendix we review some of the properties of the Lommel functions $U_n(w,s)$ and derive additional relations and limiting forms for the special case in which the arguments are of the form

$$w = \beta(\tau-|z|) \ , \tag{B1a}$$
$$s = \sqrt{\tau^2-z^2} \ , \tag{B1b}$$

with β a complex constant independent of τ and z. Below we list some properties which we shall use to derive additional relations. We restrict ourselves to the $U_n(w,s)$ Lommel functions although similar relations exist for the $V_n(w,s)$ functions and may be found in the literature[14,25–27] along with many other properties not listed here. Using therecurrence relation for Bessel functions[27], and the defining series for Lommel functions,

$$U_n(w,s) = \sum_{m=0}^{\infty}(-1)^m\Big(\frac{w}{s}\Big)^{2m+n}J_{2m+n}(s) \ , \tag{B2}$$

one may derive the following:

$$U_n(w,s) = \Big(\frac{w}{s}\Big)^n J_n(s) - U_{n+2}(w,s) \ , \tag{B3}$$
$$\frac{\partial U_n(w,s)}{\partial s} = -\frac{s}{w}U_{n+1}(w,s) \ , \tag{B4}$$
$$\frac{\partial U_n(w,s)}{\partial w} = \frac{1}{2}U_{n-1}(w,s) + \frac{1}{2}\Big(\frac{s}{w}\Big)^2 U_{n+1}(w,s) \ . \tag{B5}$$

For the variables (w,s) as defined in Eq. (B.1) we have

$$\frac{\partial U_n(\beta w,s)}{\partial|z|} = -\frac{1}{2}\Big[\beta U_{n-1}(\beta w,s) + \frac{1}{\beta}U_{n+1}(\beta w,s)\Big] \ , \tag{B6}$$

$$\frac{\partial U_n(\beta w, s)}{\partial \tau} = -\frac{1}{2}\Big[\beta U_{n-1}(\beta w, s) - \frac{1}{\beta}U_{n+1}(\beta w, s)\Big] , \quad \text{(B7)}$$

$$\frac{\partial^2 U_n(\beta w, s)}{\partial^2 |z|^2} = \frac{1}{4}\Big[\beta^2 U_{n-2}(\beta w, s) + 2U_n(\beta w, s) + \frac{1}{\beta^2}U_{n+2}(\beta w, s)\Big] , \quad \text{(B8)}$$

$$\frac{\partial^2 U_n(\beta w, s)}{\partial^2 \tau^2} = \frac{1}{4}\Big[\beta^2 U_{n-2}(\beta w, s) - 2U_n(\beta w, s) + \frac{1}{\beta^2}U_{n+2}(\beta w, s)\Big] , \quad \text{(B9)}$$

Subtracting Eq. (B.8) from Eq. (B.9) we have

$$\frac{\partial^2 U_n(\beta w, s)}{\partial^2 \tau^2} - \frac{\partial^2 U_n(\beta w, s)}{\partial^2 |z|^2} = -U_n(\beta w, s) . \quad \text{(B10)}$$

Therefore $U_n(\beta w, s)$ is a solution of the "massive" Klein-Gordon equation (at least in the positive half-space, since $|z| > 0$).

The above properties hold for arbitrary complex w and s. We now focus on the modified functions in which w is pure imaginary ($\omega \to \imath\omega$). Introducing the notation

$$\Lambda_n(w, s) = i^{-n} U_n(iw, s) , \quad \text{(B11)}$$

with w and s given by Eq. (B.1), we consider the limit $z \to 0$ for which

$$\frac{w}{s} = \sqrt{\frac{\tau - |z|}{\tau + |z|}} \to 1 . \quad \text{(B12)}$$

For n even we have

$$\Lambda_{2n}(\tau, \tau) = -\sum_{m=1}^{n-1} J_{2m}(\tau) + \frac{1 - J_0(\tau)}{2} . \quad \text{(B13)}$$

For odd n we use an integral representation

$$\Lambda_{2n+1}(\tau, \tau) = -\sum_{m=0}^{n-1} J_{2m+1}(\tau) + \frac{1}{2}\int_0^\infty dx J_0(x) , \quad \text{(B14)}$$

or in terms of Struve functions[29],

$$\Lambda_{2n+1}(\tau, \tau) = -\sum_{m=0}^{n-1} J_{2m+1}(\tau) + \frac{1}{2}\Big\{\tau J_0(\tau) + \frac{\pi\tau}{2}\big[J_1(\tau)\mathbf{H}_0(\tau) - J_0(\tau)\mathbf{H}_1(\tau)\big]\Big\} . \quad \text{(B15)}$$

Finally we consider the limiting case of $\tau = |z|$, i.e. $s = w = 0$. Since for all $n \geq 1$ $J_n(0) = 0$, we have

$$\Lambda_0(0, 0) = 1 , \quad \text{(B16a)}$$
$$\Lambda_n(0, 0) = 0 \quad n \geq 1 . \quad \text{(B16b)}$$

While some of the properties (especially B.10) derived above are useful for the actual derivation of the Green functions, they are most useful when checking the analytic expressions by operationg on them with the differential operator

$$\partial_{tt} - \partial_{xx} + V''[\phi_k(x)] . \quad \text{(B17)}$$

APPENDIX C: NUMERICAL EVALUATION AND ASYMPTOTIC FORMS FORMODIFIED LOMMEL FUNCTIONS OF TWO VARIABLES

Numerical evaluation of the Green functions derived in section III requires an evaluation of the modified Lommel functions. Although Lommel functions of two real variables[30] and two purely imaginary variables[31] have been studied, to our knowledge no one has yet considered the modified functions. Below we present methods which are valid for w complex and s real (since we start by considering the modified functions and w may be complex, our methods also include the case of two real variables). Repesenting the first argument as βw, where $|\beta| = 1$ and w and s are real, we have for the defining series

$$\Lambda_n(\beta w, s) = \sum_{m=0}^{\infty} \left(\frac{\beta w}{s}\right)^{2m+n} J_{2m+n}(s) , \quad \text{(C1)}$$

from which we deduce the symmetries

$$\Lambda_n(-\beta w, s) = (-1)^n \Lambda_n(\beta w, s) , \quad \text{(C2a)}$$
$$\Lambda_n(\beta w, -s) = \Lambda_n(\beta w, s) . \quad \text{(C2b)}$$

From Eqs. (C.2) we see that we need only investigate the first quadrant of the s–w plane. Another relationship exists which allows us to further restrict our attention to the angular region $(0, \pi/4)$ i.e., the first octant. We obtain this property by recalling the generating function for Bessel functions[32]:

$$e^{(s/2)[\beta\kappa - \frac{1}{\beta\kappa}]} = \sum_{m=-\infty}^{\infty} (\beta\kappa)^m J_m(s) , \quad \text{(C3)}$$

where $\kappa \equiv w/s$. Using the symmetry of the Bessel functions about the origin we have,

$$\cosh\Big[\frac{s}{2}\Big(\beta\kappa - \frac{1}{\beta\kappa}\Big)\Big] = \sum_{m=-\infty}^{\infty} (\beta\kappa)^{2m} J_{2m}(s) , \quad \text{(C4a)}$$
$$\sinh\Big[\frac{s}{2}\Big(\beta\kappa - \frac{1}{\beta\kappa}\Big)\Big] = \sum_{m=-\infty}^{\infty} (\beta\kappa)^{2m+1} J_{2m+1}(s) . \quad \text{(C4b)}$$

Next we note that

$$\Lambda_n\Big(\frac{s^2}{\beta w}, s\Big) = \sum_{m=0}^{\infty} (\frac{s}{\beta w})^{2m+n} J_{2m+n}(s) , \quad \text{(C5)}$$

which leads us to

$$\sinh\Big[\frac{s}{2}\Big(\beta\kappa - \frac{1}{\beta\kappa}\Big)\Big] = \Lambda_1(\beta w, s) - \Lambda_1\Big(\frac{s^2}{\beta w}, s\Big) , \tag{C6a}$$

$$\cosh\Big[\frac{s}{2}\Big(\beta\kappa - \frac{1}{\beta\kappa}\Big)\Big] = \Lambda_0(\beta w, s) + \Lambda_0\Big(\frac{s^2}{\beta w}, s\Big) - J_0(s) . \tag{C6b}$$

From Eqs. (C.6) we see that we have a relationship which allows us to consider only the region of the first quadrant of the s–w plane in which $w/s < 1$, namely the first octant. In this region the series definition converges uniformly, however that rate of convergence is very slow when w/s approaches 1. By comparison with the geometric series we see that since $J_n(s) < 1 \ \forall\ n$, we have as an error estimate for truncation after N terms

$$R_N < \frac{\kappa^{2N}}{1-\kappa^2} , \tag{C7}$$

We note that the error estimate in Eq. (C.7) is a very crude one as it does not take into account the decaying nature of the Bessel functions, however it suffices for our calculations.

As $w/s \to 1$, the number of terms in the series needed to attain a given accuracy becomes unreasonably large. For values of $\kappa = w/s$ larger than some κ_0, we turn to an asymptotic expansion[33] of the modified Lommel functions. We begin by following Mayall's[34] procedure for obtaining an integral representation for the Lommel functions by substitution of an integral representation for the Bessel functions into the series and summing the series explicitly. We restrict ourselves to deriving expressions for Λ_0 and Λ_1. For small n the asymptotic expansion for Λ_n may be obtained from the recurrence relation for Lommel functions. The large n limit has not yet been examined.

Starting with the integral representation for Bessel functions

$$J_{2m}(s) = \frac{(-1)^m}{\pi}\int_0^\pi d\theta\ e^{is\cos(\theta)}\cos(2m\theta) , \tag{C8}$$

we have

$$\begin{aligned}
&\Lambda_0(\beta w, s)\\
&= \sum_{m=0}^{\infty}(\beta\kappa)^{2m}(-1)^m\frac{1}{\pi}\int_0^\pi d\theta\ e^{is\cos(\theta)}\cos(2m\theta) , &\text{(C9)}\\
&= \frac{1}{\pi}\int_0^\pi d\theta\frac{1+(\beta\kappa)^2\cos(2\theta)}{1+2(\beta\kappa)^2\cos(2\theta)+(\beta\kappa)^4}e^{is\cos(\theta)} , &\text{(C10)}\\
&= \frac{J_0(s)}{2}\\
&+ \frac{1-(\beta\kappa)^4}{2\pi}\int_0^\pi d\theta\frac{e^{is\cos(\theta)}}{1+2(\beta\kappa)^2\cos(2\theta)+(\beta\kappa)^4} , &\text{(C11)}\\
&= \frac{J_0(s)}{2}\\
&+ \sigma_1(\beta,\kappa)\frac{\epsilon(\beta,\kappa)}{\pi}\int_0^\pi d\theta\frac{e^{is\cos(\theta)}}{\epsilon^2(\beta,\kappa)+\cos^2(\theta)} , &\text{(C12)}
\end{aligned}$$

where

$$\epsilon(\beta,\kappa) \equiv \frac{1-(\beta\kappa)^2}{2\beta\kappa} , \tag{C13a}$$

$$\sigma_1(\beta,\kappa) \equiv \frac{1+(\beta\kappa)^2}{4\beta\kappa} , \tag{C13b}$$

and uniform convergence of the sum has been used. Similarly we may write

$$\Lambda_1(\beta w, s) = -\sigma_2(\beta,\kappa)\frac{\epsilon(\beta,\kappa)}{\pi}\frac{d}{ds}\int_0^\pi d\theta\frac{e^{is\cos(\theta)}}{\epsilon^2(\beta,\kappa)+\cos^2(\theta)} , \tag{C14}$$

with

$$\sigma_2(\beta,\kappa) \equiv \frac{1+(\beta\kappa)^2}{4} + \frac{\beta\kappa[1+\epsilon^2(\beta,\kappa)]}{2\epsilon(\beta,\kappa)} . \tag{C15}$$

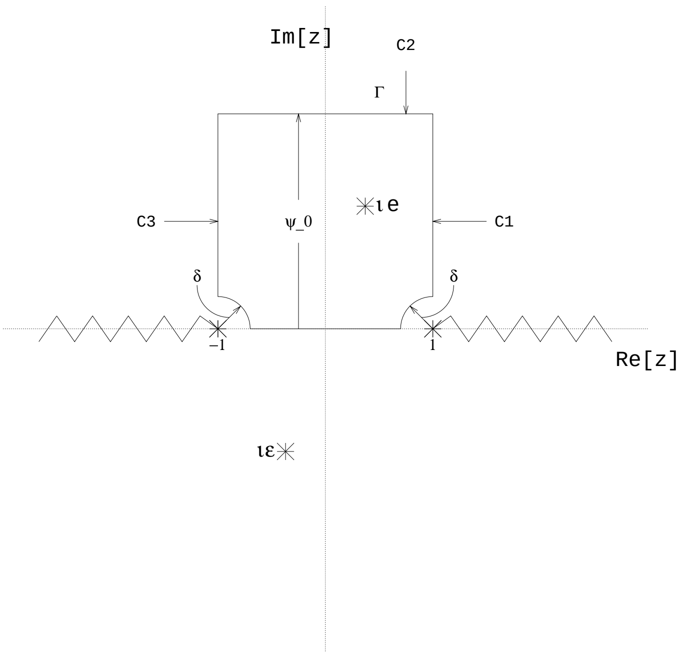
At this point, Mayall's method no longer applies (unless $\beta = \pm i$) and we turn to an alternate derivation.

The integral

$$I(\epsilon, s) = \frac{\epsilon}{\pi}\int_0^\pi d\theta\frac{e^{is\cos(\theta)}}{\epsilon^2+\cos^2(\theta)} , \tag{C16}$$

which occurs in Eqs. (C.12) and (C.14), is a strong function of ϵ since in the limit as $\epsilon \to 0$ ($w/s \to 1$), we obtain a delta function. Other major contributions occur at the stationary points $\theta = 0, \pi$. To evaluate $I(\epsilon, s)$, we substitute $t = \cos(\theta)$, deform the contour and represent the integrals as a residue which captures the strong ϵ behavior, plus two integrals for which asymptotic expansions

FIG. 3: The contour for the computation of the asymptotic expression for the modified Lommel funciton.



are easily derived. Substituting we have

$$
\begin{aligned}
&I(\epsilon, s)\\
&= \frac{\epsilon}{\pi}\int_{-1}^{1} dt \frac{e^{ist}}{(\epsilon^2+t^2)\sqrt{1-t^2}} , \qquad (C17)\\
&= \frac{\epsilon}{\pi}\Bigg\{ 2\pi i Res\big[f(z), i\epsilon\big] - \int_{c_1} dz \frac{e^{isz}}{(\epsilon^2+z^2)\sqrt{1-z^2}}\\
&\quad - \int_{c_3} dz \frac{e^{isz}}{(\epsilon^2+z^2)\sqrt{1-z^2}} \Bigg\} , \qquad (C18)
\end{aligned}
$$

where $f(z)$ is given by

$$
f(z) = \frac{e^{isz}}{(\epsilon^2+z^2)\sqrt{1-z^2}} , \qquad (C19)
$$

and the contours are shown in Figure 3. We have used the fact that as $\delta \to 0$ and $y_0 \to \infty$, the contributions from the contours $c_{\delta 1}$, $c_{\delta 2}$ and c_2 vanish by Jordon's lemma. Evaluating the residue and shifting the variables, we have

$$
\begin{aligned}
&I(\epsilon, s)\\
&= \frac{e^{-\epsilon s}}{\sqrt{1+\epsilon^2}} - \frac{\epsilon}{\pi}\int_0^{i\infty} dz \frac{e^{isz}e^{is}}{[\epsilon^2+(z+1)^2]\sqrt{1-(z+1)^2}}\\
&\quad - \frac{\epsilon}{\pi}\int_{i\infty}^{0} dz \frac{e^{isz}e^{-is}}{[\epsilon^2+(z-1)^2)]\sqrt{1-(z-1)^2}} , \qquad (C20)\\
&= \frac{e^{-\epsilon s}}{\sqrt{1+\epsilon^2}} - \frac{\epsilon}{\pi}\big[J + J^*\big] , \qquad (C21)
\end{aligned}
$$

where

$$
\begin{aligned}
J &\equiv ie^{is}\int_0^{\infty} dy \frac{e^{-sy}}{[\epsilon^2+(iy+1)^2]\sqrt{1-(iy+1)^2}} , \qquad (C22)\\
&= 2ie^{is}\int_0^{\infty} dx \frac{e^{-sx^2}}{[\epsilon^2+(ix^2+1)^2)\sqrt{x^2-2i}} . \qquad (C23)
\end{aligned}
$$

As written in Eq. (C.23), J is in one of Dingle's[35] standard integral forms which has as an asymptotic expansion

$$
J \approx 2ie^{is}\sqrt{\frac{\pi}{2F_2}}e^{-F_0}\sum_{r=0}^{\infty} Q_r , \qquad (C24)
$$

where

$$
\begin{aligned}
Q_0 &= G_0 ,\\
Q_1 &= \frac{-\sqrt{2}}{3\sqrt{\pi}F_2^{\frac{3}{2}}}\big[-3G_1F_2\big] ,\\
Q_2 &= \frac{1}{24F_2^3}\big[12G_2F_2^2\big] , \qquad (C25)\\
Q_3 &= \frac{-\sqrt{2}}{135\sqrt{\pi}F_2^{\frac{9}{2}}}\big[-45G_3F_2^3\big] ,\\
Q_4 &= \frac{1}{1152F_2^6}\big[144G_4F_2^4\big] ,\\
F_\nu &= \left(\frac{d}{dx}\right)^{\nu} sx^2 , \qquad (C26)\\
G_\nu &= \left(\frac{d}{dx}\right)^{\nu} \frac{1}{[\epsilon^2+(ix^2+1)^2]\sqrt{x^2-2i}} . \qquad (C27)
\end{aligned}
$$

Carrying out the derivatives, we have, including up to Q_4

$$
\begin{aligned}
J + J^* &= -\frac{2}{1+\epsilon^2}\sqrt{\frac{2\pi}{s}}\Bigg\{\cos(s-\frac{\pi}{4})\Big[\frac{1}{2}+\frac{R_4(\beta,\kappa)}{(8s)^2}\Big]\\
&+ \sin(s-\frac{\pi}{4})\Big[\frac{R_2(\beta,\kappa)}{(8s)}\Big]\Bigg\} + O(s^{-\frac{7}{2}}) , \qquad (C28)
\end{aligned}
$$

where

$$
\begin{aligned}
R_2(\beta,\kappa) &= \frac{9+\epsilon^2(\beta,\kappa)}{2[1+\epsilon^2(\beta,\kappa)]} , \qquad (C29)\\
R_4(\beta,\kappa) &= -\frac{9}{4}+\frac{12}{1+\epsilon^2(\beta,\kappa)}-\frac{96}{(1+\epsilon^2(\beta,\kappa))^2} \qquad (C30)
\end{aligned}
$$

With Eq. (C.28) we now have an asymptotic expansion for $I(\epsilon, s)$, which leads to the following expressions for $\Lambda_0(\beta w, s)$ and $\Lambda_1(\beta w, s)$:

$$\Lambda_0(\beta w, s) \approx \frac{J_0(s)}{2} + \sigma_1(\beta,\kappa)\frac{e^{-\epsilon(\beta,\kappa)s}}{\sqrt{1+\epsilon^2(\beta,\kappa)}} + \sigma_1(\beta,\kappa)\sqrt{\frac{2}{\pi s}}\frac{\epsilon(\beta,\kappa)}{1+\epsilon^2(\beta,\kappa)}\left\{\cos(s-\frac{\pi}{4})\Big[1+\frac{2R_4(\beta,\kappa)}{(8s)^2}\Big] + \sin(s-\frac{\pi}{4})\Big[\frac{2R_2(\beta,\kappa)}{8s}\Big]\right\} + \frac{\sigma_1(\beta,\kappa)}{\sqrt{1+\epsilon^2(\beta,\kappa)}}O(s^{-\frac{7}{2}})\ , \tag{C31}$$

$$\Lambda_1(\beta w, s) \approx \frac{\epsilon(\beta,\kappa)\sigma_2(\beta,\kappa)}{\sqrt{1+\epsilon^2(\beta,\kappa)}}\left\{e^{-\epsilon(\beta,\kappa)s} - \frac{1}{\sqrt{1+\epsilon^2(\beta,\kappa)}}\sqrt{\frac{2}{\pi s}}\Big[\cos(s-\frac{\pi}{4})\Big(\frac{2[R_2(\beta,\kappa)-2]}{8s} - 40\frac{R_4(\beta,\kappa)}{(8s)^3}\Big) - \sin(s-\frac{\pi}{4})\Big(1+\frac{2[R_4(\beta,\kappa)+12R_2(\beta,\kappa)]}{(8s)^2}\Big)\Big]\right\} + \frac{\epsilon(\beta,\kappa)\sigma_2(\beta,\kappa)}{1+\epsilon^2(\beta,\kappa)}O(s^{-\frac{9}{2}})\ , \tag{C32}$$

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[24] This result is incorrectly stated in Ref 14. Below Eq. (2) in §16.57, an error of 1/2 occurs; it should read:

$$\frac{1}{\pi}\int_0^\infty \frac{dt}{t}\sin[at+\frac{b}{t}] = J_0(2\sqrt{ab}) \tag{C33}$$

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