Parent potentials for an infinite class of reflectionless kinks

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The sine–Gordon and ϕ-four kinks are known to be reflectionless by virtue of the fact that their small oscillations are governed by the modified Pöschl–Teller potential $U_l(x) = 1 - [(l+1)/l]\mathrm{sech}^2(x/l)$, with $l = 1$ and 2, respectively. An infinite class of parent potentials $V_l(\phi)$ analogous to $V_1 \sim 1 - \cos\phi$ for sine–Gordon kinks and $V_2 \sim (1-\phi^2)^2$ for ϕ-four kinks, which bear reflectionless kinks, are constructed. This is done by requiring the lowest bound-state eigenfunction of $U_l(x)$ to be proportional to the spatial derivative of the kink waveform $\phi_K^{(l)}(x)$, i.e., the translational mode of the kink. The resulting differential equation is solved for $V_l(\phi)$ to find that it can be expressed in terms of the student's t distribution of probability theory. Various properties of the parent potentials and their reflectionless kinks are discussed.

I. INTRODUCTION

The so-called "sine–Gordon" (SG) and "ϕ-four" nonlinear wave equations have found widespread use in recent investigations of solitary-wave (kink) phenomena in both condensed matter physics[1] and particle physics.[2] These two members of the large family[2,3] of nonlinear Klein–Gordon wave equations (possessing particlelike kink solutions) share the remarkable property that they do not reflect incident linearized solutions (e.g., phonons[4]). Although this reflectionless property is not a fundamental requirement for the development of statistical mechanics phenomenologies[4–6] or perturbation theories[7–10] of kink dynamics, it does allow for a simpler treatment than for kinks without this property (e.g., the double-quadratic[5,10] and double-sine–Gordon[6] kinks).

The reflectionless property of the SG and ϕ-four kinks is found by considering the nature of small oscillations about the static kink waveforms. The spatial dependence of these small oscillations (phonons) is governed by an equation of Schrödinger form (see Sec. II below). Remarkably, the "potential" appearing in this pseudo-Schrödinger equation is of the modified Pöschl–Teller[11] type ($\sim -\mathrm{sech}^2 x$) for *both* the SG and ϕ-four cases. For special values[11] of the magnitude of this $\mathrm{sech}^2 x$ potential, incoming "particles" are not reflected but suffer only a phase shift. These special magnitudes for which the potential is reflectionless can be characterized by an integer ($l = 1,2,\ldots,\infty$); it is a curious fact that the appropriate Pöschl–Teller potentials for the SG and ϕ-four problems have $l = 1$ and 2, respectively.

The SG and ϕ-four equations arise from a nonlinear Klein–Gordon Lagrangian in which the self-coupling term ($\sim\phi^2$) is replaced by a nonlinear potential function $V(\phi)$ [$\sim 1 - \cos\phi$ for SG and $\sim(1-\phi^2)^2$ for ϕ-four; see Sec. II]. We call this potential function the "parent potential" for the kink to distinguish it from the potential function which appears in the pseudo-Schrödinger equation for the small oscillations about the kink. Noting that the parent potentials for SG and ϕ-four lead to reflectionless Pöschl–Teller potentials with $l = 1$ and 2, respectively, the question arises as to whether there might exist other parent potentials in the nonlinear Klein–Gordon family which would lead to kinks having Pöschl–Teller potentials with $l = 3$ or higher. If so, then the SG and ϕ-four kinks would be joined by other reflectionless kinks for which the statistical mechanics[3] and dynamics[7] could be treated with ease, since the eigenstates of the Pöschl–Teller potential are known exactly[11] (see Sec. III).

The search for additional parent potentials of reflectionless kinks is not merely an academic exercise. Since the SG and ϕ-four potentials cannot be expected to describe all physical situations of interest (particularly in condensed matter), it would be very beneficial to have additional model kink-bearing potentials at hand for which the small oscillations are known exactly. In this paper, we show how to *explicitly* construct the parent potentials $V_l(\phi)$ for the *entire* infinite class of reflectionless kinks whose associated small oscillations are governed by the modified Pöschl–Teller potentials with $l = 1,2,\ldots,\infty$. We find that there are two subclasses of parent potentials depending on whether l is odd or even. For l odd, the parent potentials are periodic functions of ϕ and SG is the first number ($l = 1$) of this subclass. For l even, the parent potentials have unbounded double-well character and ϕ-four is the first member ($l = 2$) of this subclass.

The outline of the remainder of the paper is as follows. In Sec. II we review the essential features of nonlinear Klein–Gordon kinks and their associated small oscillations. Special attention is paid to the zero-frequency "translation mode"[2–10] which must be present in the spectrum of small oscillations as a consequence of translational invariance. We then specialize to the SG and ϕ-four examples of parent potentials of kinks whose small oscillations are governed by the modified Pöschl–Teller (PT) potential with $l = 1$ and 2, respectively. In Sec. III we collect some of the properties of the reflectionless ($l = 1,2,3,\ldots$) PT eigenstates and give general phase-shift formulas for the continuum states (phonons). In Sec. IV we then construct the parent potentials $V_l(\phi)$ for all $l \geq 1$ by requiring the lowest bound-state eigenfunction of the PT potential to be proportional to the spatial derivative of the kink waveform $\phi_K(x)$, i.e., the translation mode of the kink. Although this identification has been noted earlier by Christ and Lee,[12] we actually carry out the explicit construc-

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tion of $V_l(\phi)$ by solving the resulting differential equation for $V_l(\phi)$. We find that the parent potential can be expressed in terms of the student's t distribution of probability theory. Plots of $V_l(\phi)$ are presented for l values up to six. The analytic properties of $V_l(\phi)$ near its degenerate minima are discussed. Finally, in Sec. V we summarize our results and discuss various uses of these higher-order parent potentials.

II. NONLINEAR KLEIN–GORDON KINKS AND THEIR SMALL OSCILLATIONS

In this section we briefly review the derivation of single-kink solutions to the equations of motion of the nonlinear Klein–Gordon variety (e.g., SG and ϕ-four) and the equation of motion for small oscillations about the kinks. We then specialize to the SG and ϕ-four cases as the two well-known examples of reflectionless kinks which have the modified Pöschl–Teller (PT) potential for their small oscillations.

The general nonlinear Klein–Gordon Lagrangian we consider has the dimensionless form

$$L = \int dx \left\{ \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - V(\phi) \right\}, \tag{2.1}$$

where x and t are dimensionless space and time variables, respectively. The nonlinear parent potential $V(\phi)$ is assumed to have at least two degenerate absolute minima, at say ϕ_1 and ϕ_2, such that $V(\phi_1) = V(\phi_2) = 0$. In addition, we assume that $V(\phi)$ is scaled in such a way that it has unit positive curvature at its degenerate minima. The nonlinear wave equation satisfied by $\phi(x,t)$ is

$$\phi_{tt} - \phi_{xx} + V'(\phi) = 0, \tag{2.2}$$

where the prime on $V(\phi)$ denotes a derivative with respect to ϕ. Static single-kink solutions $\phi_K(x)$ of Eq. (2.2) may be obtained by direct integration with the boundary conditions

$$\left.\frac{d\phi_K(x)}{dx}\right|_{x=\pm\infty} = 0 \tag{2.3a}$$

and

$$\phi_K(x = -\infty) = \phi_1, \quad \phi_K(x = +\infty) = \phi_2 \tag{2.3b}$$

for kinks and Eq. (2.3b) with $\phi_1 \leftrightarrow \phi_2$ for antikinks (this is the standard convention if $\phi_1 < \phi_2$). The kink (+) and antikink (−) solutions are given by

$$x = \pm \frac{1}{\sqrt{2}} \int_{\phi_K(0)}^{\phi_K(x)} \frac{d\phi}{\sqrt{V(\phi)}}. \tag{2.4}$$

Moving kink solutions can be obtained by a Lorentz boost. We shall henceforth be concerned only with the static kink (+) solution since it is not necessary to consider moving kinks in order to derive the general parent potentials of interest (Sec. IV).

The equation governing small oscillations about the static kink waveform is obtained by substituting

$$\phi(x,t) = \phi_K(x) + \psi(x,t) \tag{2.5}$$

into Eq. (2.2) and linearizing in ψ:

$$\psi_{tt} - \psi_{xx} + V''(\phi_K(x))\psi = 0. \tag{2.6}$$

Here $V''(\phi_K(x))$ denotes the second derivative of $V(\phi)$ with respect to ϕ evaluated for $\phi = \phi_K(x)$. Writing ψ as

$$\psi(x,t) = f(x)e^{-i\omega t} \tag{2.7}$$

leads to the following eigenvalue equation:

$$-f_{xx} + V''(\phi_K(x))f = \omega^2 f. \tag{2.8}$$

Due to the localized nature of the kink waveform $\phi_K(x)$, the function $V''(\phi_K(x))$ varies mainly in the region near the kink center (assumed to be at $x = 0$) and approaches unity (due to our assumption of unit curvature) far away from the kink center,

$$V''(\phi_K(x)) \underset{|x|\to\infty}{\to} 1. \tag{2.9}$$

Moreover, the function $V''(\phi_K(x))$ has a minimum at $x = 0$ such that

$$V''(\phi_K(x)) < 0. \tag{2.10}$$

From these properties, we see that there exists a close analogy between Eq. (2.8) and the Schrödinger equation for a particle moving in a one-dimensional "potential well," $V''(\phi_K(x))$. The "bound state(s)" and "continuum" states for this potential are of *fundamental* importance for statistical mechanics phenomenologies,[4] perturbation theories of kink dynamics,[7] and quantization procedures for kink states.[2,12–17]

Since the Lagrangian (2.1) possesses translation invariance, the spectrum of small oscillations about a single kink must contain a zero-frequency ($\omega = 0$) translation mode (Goldstone mode) that restores the translation invariance broken by the introduction of a kink at $x = 0$. This means that Eq. (2.8) must *always* possess a "bound" state solution with $\omega^2 = \omega_{b,1}^2 = 0$ (and perhaps other bound states with $0 < \omega^2 < 1$) and the corresponding bound state wave function $f_{b,1}(x)$ will be proportional to the spatial derivative of $\phi_K(x)$,

$$f_{b,1}(x) \propto \frac{d\phi_K(x)}{dx}, \tag{2.11}$$

as can be shown easily by differentiating Eq. (2.2) with respect to x and setting $\phi = \phi_K(x)$.

In addition to the translation mode at $\omega^2 = 0$, there may exist additional bound state solutions of Eq. (2.8) with nonzero frequencies (between 0 and 1). The solutions correspond to "internal" oscillation modes in which the kink waveform undergoes a harmonically varying shape change localized about the kink center. We denote the bound-state eigenfrequencies by $\omega_{b,1} = 0, \omega_{b,2},\ldots,\omega_{b,N_b}$, where N_b is the total number of bound states. The lowest of these is $\omega_{b,1} = 0$ (the translation mode) since all other $\omega_{b,n}^2$ must be nonnegative in order for the kink to be stable against small oscillations.

In addition to the bound-state solutions of Eq. (2.8), there exist continuum states (extended modes) which we label by a wave vector k. These states have eigenvalues ω_k^2 given by

$$\omega_k^2 = 1 + k^2, \tag{2.12}$$

which is precisely the dispersion relation for small oscillations (phonons) in the *absence* of kinks. Equation (2.12) follows from the fact that far away from the kink the potential $V''(\phi_K(x))$ approaches unity [Eq. (2.9)]. Although the precise form of the continuum eigenfunction $f_k(x)$ can be quite complicated in the region of the kink, it has the following simple asymptotic form for the reflectionless potentials $V''(\phi_K(x))$ which we consider in this paper:

$$f_k(x) \underset{x\to\pm\infty}{\rightarrow} A_k \exp i[kx \pm \tfrac{1}{2}\Delta(k)], \qquad (2.13)$$

where $\Delta(k)$ is a phase-shift function which depends on the potential at hand. This phase-shift function is an extremely important quantity since it contains all the information concerning kink–phonon interactions that is needed to renormalize the kink energy due to thermal[4] or quantum[14] fluctuations (or both[17]).

Now we consider the SG and ϕ-four cases in particular. The parent potentials are

$$V_1(\phi) = 1 - \cos\phi \quad \text{(SG)}, \qquad (2.14a)$$

$$V_2(\phi) = \tfrac{1}{8}(\phi^2 - 1)^2 \quad (\phi\text{-four}), \qquad (2.14b)$$

where the meaning of the subscripts 1 and 2 will become clear in a moment. The static single-kink solutions are

$$\phi_K(x) = 4\tan^{-1} e^x \quad \text{(SG)}, \qquad (2.15a)$$

$$\phi_K(x) = \tanh(x/2) \quad (\phi\text{-four}). \qquad (2.15b)$$

For future reference we give the dimensionless energies of the static kinks obtained from the general relation,[4]

$$E_K = 2\int_{-\infty}^{+\infty} dx\, V(\phi_K(x)) = \sqrt{2}\int_{\phi_1}^{\phi_2} d\phi \sqrt{V(\phi)}\,. \qquad (2.16)$$

These are

$$E_K = 8 \quad \text{(SG)}, \qquad (2.17a)$$

$$E_K = \tfrac{2}{3} \quad (\phi\text{-four}). \qquad (2.17b)$$

From Eqs. (2.14) and (2.15) it is straightforward to derive the potential function $V''(\phi_K(x))$ which appears in Eq. (2.8). We find

$$V_1''(\phi_K(x)) = 1 - 2\,\text{sech}^2 x \quad \text{(SG)}, \qquad (2.18a)$$

$$V_2''(\phi_K(x)) = 1 - \tfrac{3}{2}\,\text{sech}^2(x/2) \quad (\phi\text{-four}). \qquad (2.18b)$$

These two potentials are special cases ($l = 1$ and 2, respectively) of the modified Pöschl–Teller (PT) potential,[11]

$$U_l(x) = 1 - [(l+1)/l]\,\text{sech}^2(x/l), \qquad (2.19)$$

for which the exact eigenstates are known analytically.[11] The SG case has exactly one bound state (the translation mode),

$$f_{b,1}(x) = (1/\sqrt{2})\,\text{sech}\, x, \quad \omega_{b,1}^2 = 0, \qquad (2.20)$$

and the continuum states

$$f_k(x) = [2\pi(1+k^2)]^{-1/2} e^{ikx}(k + i\tanh x). \quad (2.21)$$

The ϕ-four case has *two* bound states,

$$f_{b,1}(x) = \tfrac{1}{2}\sqrt{\tfrac{3}{2}}\,\text{sech}^2(x/2), \quad \omega_{b,1}^2 = 0, \qquad (2.22a)$$

$$f_{b,2}(x) = \tfrac{1}{2}\sqrt{3}\,\text{sech}(x/2)\tanh(x/2), \quad \omega_{b,2}^2 = \tfrac{3}{4}, \qquad (2.22b)$$

and the continuum states

$$f_k(x) = [8\pi(1+k^2)(1+4k^2)]^{-1/2} e^{ikx} \times (3\tanh^2(x/2) - 6ik\tanh(x/2) - [1+4k^2])\,. \qquad (2.23)$$

These states satisfy the orthonormality conditions

$$\int_{-\infty}^{+\infty} dx\, f_{b,n}(x) f_{b,m}(x) = \delta_{n,m}\,, \qquad (2.24a)$$

$$\int_{-\infty}^{+\infty} dx\, f_k^*(x) f_{k'}(x) = \delta(k-k'), \qquad (2.24b)$$

$$\int_{-\infty}^{+\infty} dx\, f_k(x) f_{b,n}(x) = 0, \qquad (2.24c)$$

and the completeness relation,

$$\sum_{n=1}^{N_b} f_{b,n}(x) f_{b,n}(x') + \int_{-\infty}^{+\infty} dk\, f_k^*(x) f_k(x') = \delta(x-x'). \qquad (2.25)$$

From Eqs. (2.21) and (2.23) we find the phase shift function defined in Eq. (2.13):

$$\Delta_1(k) = \pi(k/|k|) - 2\tan^{-1}k \quad \text{(SG)}, \qquad (2.26a)$$

$$\Delta_2(k) = 2\pi(k/|k|) - 2\tan^{-1}k - 2\tan^{-1}2k \quad (\phi\text{-four}). \qquad (2.26b)$$

III. EIGENSTATES OF THE REFLECTIONLESS MODIFIED PÖSCHL–TELLER POTENTIAL

In this section we collect some of the useful properties of the eigenstates of the modified PT potential (2.19) for arbitrary positive values of the integer index l. In the next section we shall construct the parent potentials $V_l(\phi)$ for which the small oscillations about the kink solutions satisfy Eq. (2.8) with the PT potential,

$$V_l''(\phi_K(x)) = 1 - [(l+1)/l]\,\text{sech}^2(x/l), \qquad (3.1)$$

namely,

$$-\frac{d^2 f^{(l)}}{dx^2} + \left[1 - \frac{l+1}{l}\,\text{sech}^2\frac{x}{l}\right] f^{(l)} = \omega^2 f^{(l)}\,. \qquad (3.2)$$

Using the substitution $\eta = \tanh(x/l)$, this equation becomes the associated Legendre equation[18]

$$(1-\eta^2)\frac{d^2 f^{(l)}}{d\eta^2} - 2\eta\frac{df^{(l)}}{d\eta} + \left[l(l+1) - \frac{l^2(1-\omega^2)}{1-\eta^2}\right] f^{(l)} = 0\,. \qquad (3.3)$$

The normalized bound state solutions are

$$f_{b,n}^{(l)}(x) = \left\{\frac{(l-n+1)(n-1)!}{l(2l-n+1)!}\right\}^{1/2} P_l^{l-n+1}\left(\tanh\frac{x}{l}\right) \quad (n = 1,2,\ldots,l)\,, \qquad (3.4)$$

with eigenvalues

$$\omega_{b,n}^2 = 1 - [1 - [(n-1)/l]]^2\,. \qquad (3.5)$$

The function $P_l^{l-n+1}(\tanh(x/l))$ is the associated Legendre polynomial.[18] The continuum states are

$$f_k^{(l)}(x) = A_k^{(l)} P_l^{ilk}(\tanh(x/l)), \qquad (3.6)$$

where $A_k^{(l)}$ is a normalization constant which we will not need for our purposes. Using the expression of $P_\mu^\nu(z)$ in terms of the hypergeometric function,[19] we can write

$$f_k^{(l)}(x) = \frac{A_k^{(l)}}{\Gamma(1-ilk)} \times e^{ikx} F(-l,\ l+1;\ 1-ilk;\ \tfrac{1}{2}[1-\tanh(x/l)]) . \tag{3.7}$$

This hypergeometric function F is simply a polynomial of degree l in the variable, $1 - \tanh x/l$. From its asymptotic behavior[20] we can calculate the phase shift function $\Delta_l(k)$

$$\Delta_l(k) = l\pi \frac{k}{|k|} - 2\sum_{n=1}^{l} \tan^{-1}\left(\frac{lk}{n}\right). \tag{3.8}$$

The only state which we shall need in order to construct these parent potentials is the ground state (translation mode),

$$f_{b,1}^{(l)}(x) = [(2l-1)!!/2^l l!]^{1/2} \operatorname{sech}^l(x/l). \tag{3.9}$$

IV. CONSTRUCTION OF THE PARENT POTENTIALS

We seek the parent potentials $V_l(\phi)$ whose daughter kinks $\phi_K(x)$ are reflectionless by virtue of the potentials they present to small oscillations,

$$V_l''(\phi_K(x)) \equiv 1 - [(l+1)/l] \operatorname{sech}^2(x/l). \tag{4.1}$$

From Eq. (2.11) we know that the translation mode $f_{b,1}^{(l)}(x)$ [Eq. (3.9)] is proportional to $d\phi_K(x)/dx$. Let the proportionality constant be denoted by α_l; it will be chosen later in a convenient manner. Thus we have

$$f_{b,1}^{(l)}(x) = \alpha_l \frac{d\phi_K(x)}{dx}. \tag{4.2}$$

From Eq. (2.4), $d\phi_K/dx$ can be expressed in terms of V_l itself,

$$\frac{d\phi_K}{dx} = [2\, V_l(\phi_K(x))]^{1/2}. \tag{4.3}$$

Combining Eqs. (3.9), (4.2), and (4.3), we have

$$\operatorname{sech}^l(x/l) = \alpha_l [2^l l!/(2l-1)!!]^{1/2} [2\, V_l(\phi_K(x))]^{1/2}. \tag{4.4}$$

Substitution of Eq. (4.4) into Eq. (4.1) then yields

$$V_l''(\phi_K(x)) = 1 - [(l+1)/l] [V_l(\phi_K(x))/V_l^0]^{1/l}, \tag{4.5}$$

where

$$V_l^0 \equiv (1/\alpha_l^2)[(2l-1)!!/2^{l+1} l!] \tag{4.6}$$

is the height of the barrier between adjacent minima (at ϕ_1 and ϕ_2) in the parent potential, as can be seen by setting $x=0$ in Eq. (4.4).

Equation (4.5) provides a differential equation for $V_l(\phi)$ for all values of ϕ between ϕ_1 and ϕ_2 [the range swept by the kink $\phi_K(x)$]. Thus

$$\frac{d^2 V_l(\phi)}{d\phi^2} = 1 - \frac{l+1}{l}\left[\frac{V_l(\phi)}{V_l^0}\right]^{1/l} \quad (\phi_1 < \phi < \phi_2). \tag{4.7}$$

We remark that the positive real branch of the lth root of $V_l(\phi)/V_l^0$ must be chosen in Eq. (4.7), to be consistent with Eq. (4.4) for $\phi_1 \leqslant \phi \leqslant \phi_2$. Although Eq. (4.7) was derived assuming ϕ is in the kink range, we may obtain $V_l(\phi)$ for ϕ outside this range by an appropriate analytic continuation. It is convenient to proceed separately for the cases l odd and l even.

(i) l odd: All of the parent potentials in this class are periodic functions of ϕ with period $\phi_2 - \phi_1$; they have the same topology as SG ($l=1$) which is the first member of the class. It is therefore convenient for l odd to choose $\phi_1 = 0$ and $\phi_2 = 2\pi$ so that all of these potentials have period 2π. This choice fixes the barrier height V_l^0, which in turn determines the constant α_l via Eq. (4.6). Furthermore, we see from Eq. (4.7) that $V_l(\phi)$ can be made symmetric in ϕ,

$$V_l(-\phi) = V_l(\phi). \tag{4.8}$$

Combining this symmetry with the periodicity property, we see that $V_l(\phi)$ is also symmetric about π. Thus we need only integrate Eq. (4.7) in the range $0 < \phi < \pi$. The first integral is easily obtained,

$$\frac{dV_l(\phi)}{d\phi} = [2V_l(\phi)]^{1/2}\left\{1 - \left[\frac{V_l(\phi)}{V_l^0}\right]^{1/l}\right\}^{1/2} \tag{4.9}$$

Integrating once more, we have

$$\phi = \frac{1}{\sqrt{2}} \int_0^{V_l(\phi)} \frac{dV}{\sqrt{V}} \left\{1 - \left[\frac{V}{V_l^0}\right]^{1/l}\right\}^{-1/2}. \tag{4.10}$$

This equation gives an implicit solution for $V_l(\phi)$ for $0 < \phi < \pi$, and can be reexpressed in several equivalent forms. Defining

$$\xi_l = [V_l(\phi)/V_l^0]^{1/2l}, \tag{4.11}$$

we can write

$$\phi = l(2V_l^0)^{1/2} \int_0^{\sin^{-1}\xi_l} d\theta \sin^{l-1}\theta \tag{4.12}$$

$$= l(2V_l^0)^{1/2} \int_0^{\xi_l} dy\, y^{l-1}(1-y^2)^{-1/2} \tag{4.13}$$

$$= l\left(\frac{V_l^0}{2}\right)^{1/2} B\left(\frac{l}{2},\frac{1}{2}\right) I_{\xi_l^2}\left(\frac{l}{2},\frac{1}{2}\right), \tag{4.14}$$

where $I_x(a,b)$ is the normalized incomplete beta function[21] and $B(a,b)$ is the beta function.[22] Here $B(a,b)$ may be expressed[22] in terms of gamma functions by

$$B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b). \tag{4.15}$$

The function $I_x(a,b)$ lies in the unit interval: $I_0(a,b) = 0$, $I_1(a,b) = 1$. The barrier height V_l^0 is now determined by noting that $\phi = \pi$ is the midpoint of the barrier and $\xi_l = 1$ at this point. Thus from Eq. (4.14) we have

$$V_l^0 = 2\pi^2/l^2 B^2(l/2,\tfrac{1}{2}) \quad (l \text{ odd}). \tag{4.16}$$

Equation (4.14) then becomes

$$\phi = \pi I_{\xi_l^2}(l/2,\tfrac{1}{2}). \tag{4.17}$$

Because of the particular values of the indices of the incomplete beta function appearing in Eq. (4.17), we can relate $I_{\xi_l^2}(l/2,\frac{1}{2})$ to the so-called "student's t distribution,"[21] $A(t\,|l)$ which arises in the theory of probability distribution functions.[21] Namely,

$$I_{\xi_l^2}(l/2,\tfrac{1}{2}) = 1 - A(t\,|l), \tag{4.18}$$

where

$$t \equiv \sqrt{l}\,[(\sqrt{1-\xi_l^2})/\xi_l]. \tag{4.19}$$

Thus

$$1-\phi/\pi = A(t\,|\,l) \quad (0<\phi<\pi) \quad (l \text{ odd}). \tag{4.20}$$

If we denote the inverse of the student's t distribution function by $A_l^{-1}(x)$, then

$$t = A_l^{-1}(1-\phi/\pi). \tag{4.21}$$

From Eq. (4.19), we have

$$\xi_l^2 = l/(l+t^2). \tag{4.22}$$

Hence, using Eqs. (4.11), (4.21), and (4.22), we finally obtain a formal expression for the normalized parent potential $\widetilde{V}_l(\phi) \equiv V_l(\phi)/V_l^0$,

$$\widetilde{V}_l(\phi) = \{l/(l+[A_l^{-1}(1-\phi/\pi)]^2)\}^l$$

$$(l \text{ odd};\ 0<\phi<\pi). \tag{4.23}$$

For ϕ outside this range, $\widetilde{V}_l(\phi)$ is continued from Eq. (4.23) using

$$\widetilde{V}_l(2\pi-\phi) \equiv \widetilde{V}_l(\phi) = \widetilde{V}_l(\phi+2\pi n), \tag{4.24}$$

where n is any integer.

Although Eqs. (4.23) and (4.24) represent the formal solution to the problem of finding the periodic (l odd) parent potentials, it is not convenient to use Eq. (4.23) directly if one wishes, for example, to obtain a plot of $\widetilde{V}_l(\phi)$ vs ϕ. Instead, it is much easier to obtain ϕ in terms of $\widetilde{V}_l$ by making use of Eq. (4.20) and a finite series expansion[21,23] of the student's t distribution,

$$A(t\,|\,l) = \frac{2}{\pi}\left\{\theta + \sin\theta \sum_{n=1}^{(l-1)/2} \frac{(2n-2)!!}{(2n-1)!!}\cos^{2n-1}\theta\right\}, \tag{4.25}$$

where

$$\theta \equiv \tan^{-1}(t/\sqrt{l}). \tag{4.26}$$

Using Eq. (4.22) we can reexpress θ as

$$\theta = \cos^{-1}\xi_l \tag{4.27}$$

and from Eqs. (4.20), (4.11), (4.25), and (4.26) we then have

$$\frac{\pi-\phi}{2} = \cos^{-1}\widetilde{V}_l^{1/2l} + \{1-\widetilde{V}_l^{1/l}\}^{1/2} \sum_{n=1}^{(l-1)/2} \frac{(2n-2)!!}{(2n-1)!!}\widetilde{V}_l^{(n-1/2)/l}. \tag{4.28}$$

Equation (4.28) provides a simple, explicit expression for ϕ as a function of $\widetilde{V}_l$ which may be plotted in the range $0<\phi<\pi$, $0<\widetilde{V}_l<1$. The curve thus obtained can then be inverted and extended to include several periods of $\widetilde{V}_l(\phi)$ using Eq. (4.24) if desired. In Fig. 1, we show the results of this procedure for $l=3$ and 5, in addition to a plot of $\widetilde{V}_1(\phi) = \frac{1}{2}(1-\cos\phi)$, which is the sine–Gordon potential. Note the tendency for the barrier between adjacent minima to become more plateaulike as l is increased. Indeed, this behavior becomes extreme in the limit as $l\to\infty$. This can be seen from the limit of the student's t distribution,

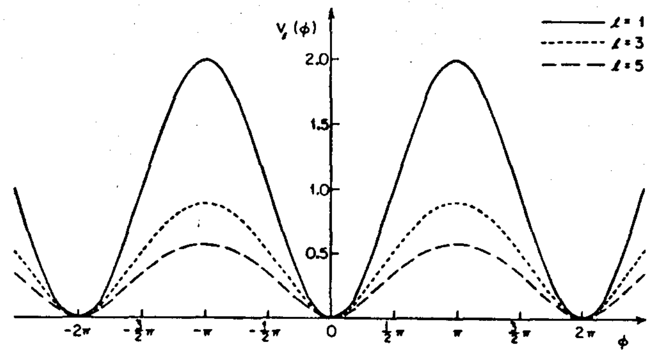

FIG. 1. The first three periodic parent potentials (l odd).

$$\lim_{l\to\infty} A(t\,|\,l) \equiv A(t) = \frac{1}{\sqrt{2\pi}}\int_{-t}^{t} e^{-x^2/2}\,dx$$

$$= \operatorname{erf}(\sqrt{2}\,t), \tag{4.29}$$

where $\operatorname{erf}(\sqrt{2}\,t)$ is the error function.[24] Thus using Eqs. (4.11), (4.19), and (4.20), we have in the limit $l\to\infty$

$$\phi/\pi = \lim_{l\to\infty}\operatorname{erfc}[\sqrt{2l}\{(1-\widetilde{V}_l^{1/l})/\widetilde{V}_l^{1/l}\}^{1/2}], \tag{4.30}$$

where $\operatorname{erfc} z = 1-\operatorname{erf} z$ is the complimentary error function.[24] Since this function becomes very sharply peaked as a function of $[(1-\widetilde{V}_l^{1/l})/\widetilde{V}_l^{1/l}]^{1/2}$ when l becomes large, we see that $\widetilde{V}_l^{1/l}$ (and hence $\widetilde{V}_l$) must remain very close to unity as ϕ is decreased from π until ϕ nears zero at which point $\widetilde{V}_l$ must drop sharply to zero. We note, however, that the barrier height V_l^0 tends to zero in the large l limit as

$$V_l^0 \underset{l\to\infty}{\to} \pi/l \tag{4.31}$$

as can be shown from Eqs. (4.15) and (4.16) and the use of Sterling's asymptotic formula[22] for the gamma function.

(ii) l even: All of the parent potentials in this class have the same topology as the ϕ-four potential ($l=2$), namely a double-well structure. It is therefore convenient for l even to

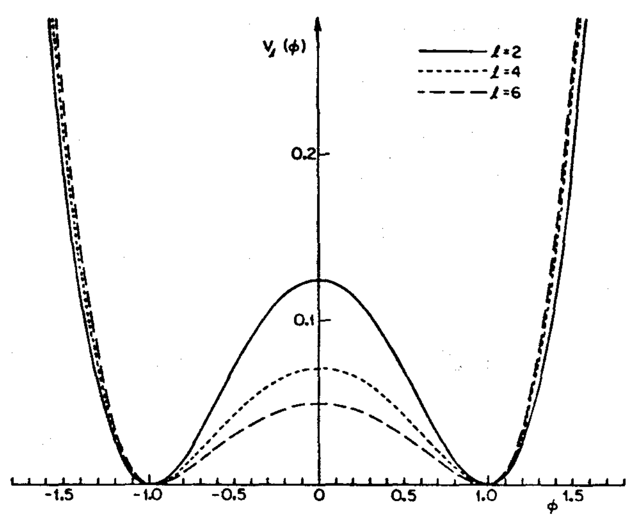

FIG. 2. The first three double-well parent potentials (l even). For $\phi\simeq 2.1$, the curves cross so that for fixed $\phi_0 \gtrsim 2.1$, $V_2(\phi_0) > V_4(\phi_0) > V_6(\phi_0)$.

choose $\phi_1 = -1$ and $\phi_2 = +1$ so that all of these potentials have their two degenerate well-minima located at $\phi = \pm 1$. With this choice the potential is symmetric about $\phi = 0$, where the barrier has its maximum V_l^0. Thus to first obtain $V_l(\phi)$ in the kink range, $-1<\phi<+1$, we need only integrate Eq. (4.7) in the range $0<\phi<1$. The slope of $V_l(\phi)$ is negative in this range, so that the first integral of Eq. (4.7) is

$$\frac{dV_l(\phi)}{d\phi} = -[2V_l(\phi)]^{1/2}\left\{1-\left[\frac{V_l(\phi)}{V_l^0}\right]^{1/l}\right\}^{1/2} \tag{4.32}$$

Integrating once more we have

$$\phi = -\frac{1}{\sqrt{2}}\int_{V_l^0}^{V_l(\phi)} \frac{dV}{\sqrt{V}}\left\{1-\left(\frac{V}{V_l^0}\right)^{1/l}\right\}^{-1/2} \tag{4.33}$$

$$= -l(2V_l^0)^{1/2}\int_1^{\xi_l} dy\, y^{l-1}(1-y^2)^{-1/2}$$

$$= l(2V_l^0)^{1/2}\left[\int_0^1 dy\, y^{l-1}(1-y^2)^{-1/2} - \int_0^{\xi_l} dy\, y^{l-1}(1-y^2)^{-1/2}\right] \tag{4.34}$$

$$= l\left[\frac{V_l^0}{2}\right]^{1/2} B\left(\frac{l}{2},\frac{1}{2}\right)\left[1-I_{\xi_l^2}\left(\frac{l}{2},\frac{1}{2}\right)\right]. \tag{4.35}$$

Noting that when $\phi = 1$, ξ_l^2 must equal zero [Eq. (4.11)], we then obtain an expression for the barrier height when l is even,

$$V_l^0 = 2/l^2B^2(l/2,\tfrac{1}{2}) \quad (l \text{ even}). \tag{4.36}$$

Equation (4.35) then becomes

$$1-\phi = I_{\xi_l^2}(l/2,\tfrac{1}{2}) \quad (0<\phi<1), \tag{4.37}$$

or, using the student's t distribution,

$$\phi = A(t\,|\,l). \tag{4.38}$$

This may be formally inverted as for the odd l case to give the normalized parent potential

$$\widetilde{V}_l(\phi) = \{l/(l+[A_l^{-1}(\phi)]^2)\}^l \quad (l \text{ even};\ 0<\phi<1). \tag{4.39}$$

Equation (4.38) can be made more explicit by using the series expansion for $A(t\,|\,l)$ when l is even[21]:

$$A(t\,|\,l) = \sin\theta\left\{1+\sum_{n=1}^{l/2-1}\frac{(2n-1)!!}{(2n)!!}\cos^{2n}\theta\right\}, \tag{4.40}$$

where θ is given in terms of t by Eq. (4.26) and in terms of ξ_l by Eq. (4.27). Thus Eq. (4.38) can be rewritten as

$$\phi = \{1-\widetilde{V}_l^{1/l}\}^{1/2}\left[1+\sum_{n=1}^{l/2-1}\frac{(2n-1)!!}{(2n)!!}\widetilde{V}_l^{n/l}\right]$$

$$(l \text{ even};\ 0<\phi<1), \tag{4.41}$$

where we have used Eq. (4.11).

To obtain results for $|\phi|>1$, we return to Eq. (4.32), where we note that if $V_l(\phi)\to\infty$ as $|\phi|\to\infty$ for each l, the slope becomes imaginary when $\widetilde{V}_l>1$. Choosing the negative real root of $(\widetilde{V}_l)^{1/l}$ avoids this problem and exactly reproduces the ϕ-four solution (for $l=2$). Since $V(\phi) = V(-\phi)$ we restrict our attention to $\phi>1$ where the slope $dV_l/d\phi>0$ and write

$$\frac{dV_l}{d\phi} = \sqrt{2V_l(\phi)}\left[1+\left(\frac{V_l(\phi)}{V_l^0}\right)^{1/l}\right]^{1/2} \tag{4.42}$$

Hence

$$\phi - 1 = \frac{1}{\sqrt{2}}\int_0^{V_l(\phi)}\frac{dV}{\sqrt{V}}\left[1+\left(\frac{V}{V_l^0}\right)^{1/l}\right]^{-1/2} \tag{4.43}$$

$$= l\sqrt{2V_l^0}\int_0^{\sinh^{-1}\xi_l}(\sinh x)^{l-1}\,dx \tag{4.44}$$

$$= l\sqrt{2V_l^0}(-1)^{(l-2)/2}\sum_{n=0}^{(l-2)/2}(-1)^n\binom{\frac{l-2}{2}}{n} \times\frac{1}{2n+1}\left[(\xi_l^2+1)^{(2n+1)/2}-1\right]. \tag{4.45}$$

Therefore

$$\phi = 1 + l\sqrt{2V_l^0}(-1)^{(l-2)/2} \times\sum_{n=0}^{(l-2)/2}(-1)^n\binom{(l-2)/2}{n} \times\frac{1}{2n+1}\left\{\left[\left(\frac{V_l}{V_l^0}\right)^{1/l}+1\right]^{(2n+1)/2}-1\right\}$$

$$(l \text{ even};\ \phi>1). \tag{4.46}$$

Equations (4.41) and (4.46) can be used to obtain ϕ vs $\widetilde{V}_l(\phi)$ plots for all $\phi>0$, whereupon the plots can be inverted to give $\widetilde{V}_l(\phi)$. For $\phi<0$, we use the symmetry of $\widetilde{V}_l$: $\widetilde{V}_l(-\phi) = \widetilde{V}_l(\phi)$. In Fig. 2 we show the results of this procedure for $l=4$ and 6, as well as the ϕ-four parent potential for comparison.

Apart from the ϕ-four case ($l=2$), the only other case for which Eqs. (4.41) and (4.46) can be inverted analytically is $l=4$. In this case, they become third degree polynomials in $\xi_4^2 = \widetilde{V}_4^{1/4}$ whose roots can be found in closed form[25]:

$$\widetilde{V}_4(\phi) = \begin{cases}[2\cos(\tfrac{1}{3}\sin^{-1}\phi)-1]^4, & 0<|\phi|<1,\\ [\cos(\tfrac{1}{3}\sin^{-1}(\phi-2))+\sqrt{3}\sin(\tfrac{1}{3}\sin^{-1}(\phi-2))+1]^4, & 1<|\phi|<3,\\ [2\cosh(\tfrac{1}{3}\cosh^{-1}(\phi-2))+1]^4, & |\phi|>3.\end{cases} \tag{4.47}$$

TABLE I. Barrier heights and kink rest energies for the first six parent potentials.

l	V_l^0	$E_K^{(l)}$
1	2	8
2	$\frac{1}{4}$	$\frac{2}{3}$
3	$\frac{1}{9}$	$\frac{256}{45}$
4	$\frac{9}{128}$	$\frac{18}{35}$
5	$\frac{128}{225}$	$\frac{65\,536}{14\,175}$
6	$\frac{25}{512}$	$\frac{100}{231}$

From Eqs. (2.16) and (4.9), we can obtain a simple expression for the kink rest energies, $E_K^{(l)}$, for odd l,

$$E_K^{(l)} = \sqrt{2}\int_{\phi_1}^{\phi_2} d\phi [V_l(\phi)]^{1/2}$$

$$= 2\sqrt{2}\int_0^{V_l^0} dV \left(\frac{dV}{d\phi}\right)^{-1} \sqrt{V}$$

$$= 2lV_l^0 \int_0^1 dz\, z^{l-1}(1-z)^{-1/2}$$

or

$$E_K^{(l)} = 2lV_l^0 B(l,\tfrac{1}{2}). \tag{4.48}$$

This result also holds for l even. Using Eqs. (4.15), (4.16), and (4.36), we have calculated the parent potential barrier heights V_l^0 and kink rest energies for the first several values of l and for convenience listed these in Table I.

The coefficients α_l in Eq. (4.2) can be shown, using Eqs. (4.6) and (4.48), to be simply related to the kink rest energies via

$$\alpha_l = [E_K^{(l)}]^{-1/2}. \tag{4.49}$$

The actual static kink waveforms can be obtained from Eqs. (4.2), (4.6), and (3.9) in a straightforward manner,

$$\frac{d\phi_K^{(l)}(x)}{dx} = \alpha_l^{-1} f_{b,1}^{(l)}(x)$$

$$= (2V_l^0)^{1/2}\,\mathrm{sech}^l(x/l). \tag{4.50}$$

Thus

$$\phi_K^{(l)}(x) = l(2V_l^0)^{1/2}\int^{x/l} dy\, \mathrm{sech}^l y, \tag{4.51}$$

where the lower limit on the integration is unnecessary since its contribution has been cancelled by ϕ_1 on the left-hand side. The indefinite integral of $\mathrm{sech}^l y$ can be found in the tables of Ref. 26. We present explicit forms for l up to 6 in Table II and in Figs. 3 and 4 we plot the waveforms for these six reflectionless kinks.

TABLE II. Static waveforms for the first six reflectionless kinks.

l	$\phi_K^{(l)}(x)$
1	$4\tan^{-1} e^x$
2	$\tanh(x/2)$
3	$4\tan^{-1} e^{x/3} + 2\,\mathrm{sech}(x/3)\tanh(x/3)$
4	$\frac{3}{2}\tanh(x/4)[1-\frac{1}{3}\tanh^2(x/4)]$
5	$4\tan^{-1} e^{x/5} + [\frac{4}{3}\mathrm{sech}^3(x/5) + 2\,\mathrm{sech}(x/5)]\tanh(x/5)$
6	$\frac{15}{8}\tanh(x/6)[1-\frac{2}{3}\tanh^2(x/6)+\frac{1}{5}\tanh^4(x/6)]$

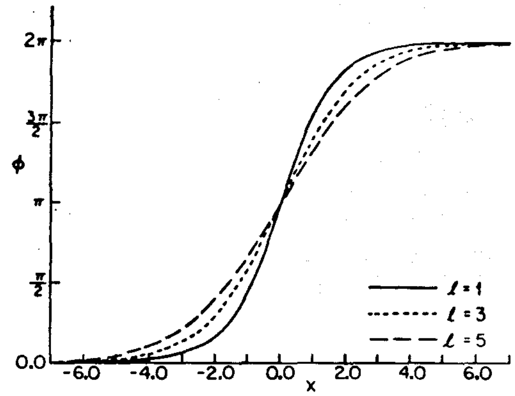

FIG. 3. Static kink waveforms for $l = 1$ (SG), 3, and 5.

The analytic properties of the parent potential $V_l(\phi)$ can be exhibited by considering its successive derivatives near one of its degenerate minima. By construction, the first derivative vanishes at ϕ_1 or ϕ_2, and the second derivative approaches unity at these values. The third derivative can be obtained from Eqs. (4.7) and (4.9) (or 4.32),

$$\frac{d^3V_l(\phi)}{d\phi^3} = \pm\sqrt{2}\,\frac{l+1}{l^2}\,[V_l^0]^{-1/l}[V_l(\phi)]^{1/l-1/2} \times\left\{1-\left[\frac{V_l(\phi)}{V_l^0}\right]^{1/l}\right\}^{1/2}. \tag{4.52}$$

As ϕ approaches a potential minimum from small $|\phi|$ values, $V_l(\phi)$ approaches zero, so that

$$\frac{d^3V_l(\phi)}{d\phi^3} \underset{\phi\to\phi_{1,2}}{\rightarrow} \pm\sqrt{2}\,\frac{l+1}{l^2}\,[V_l^{(0)}]^{-1/l}[V_l(\phi)]^{1/l-1/2} \tag{4.53}$$

and we see that for $l > 2$, the third (and higher) derivatives are singular at the well minima. (This fact has been noted previously by Christ and Lee[12].) Thus the parent potentials for $l > 2$ do not possess Taylor expansions about their minima. Although the parent potentials and their first two de-

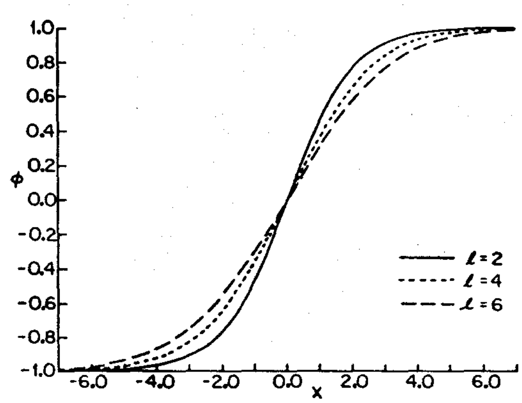

FIG. 4. Static kink waveforms for $l = 2$ (ϕ-four), 4, and 6.

rivatives are continuous, the singularities in their higher derivatives can be expected to cause difficulties in calculations which incorporate these derivatives; for example: high-order perturbation theories of kink response to external forces,[7–10] "anharmonic phonon" contributions to statistical mechanical quantities,[3–6] and quantum renormalization[2,12–17] of kink energies. Nevertheless, if one is interested only in the lowest-order "Gaussian" fluctuations about the kink solutions, the parent potentials we have constructed in this section are well-behaved to this order.

V. SUMMARY AND DISCUSSION

In this paper we have obtained the formal solution [Eqs. (4.23) and (4.29)] to the problem of finding parent potentials for an infinite class of nonlinear Klein–Gordon kinks which are reflectionless by virtue of the fact that they present a modified Pöschl–Teller potential of special magnitude to the small oscillations (e.g., phonons). We found that these parent potentials $V_l(\phi)$ fall into two subclasses: for l odd they are periodic functions of the field ϕ and the sine–Gordon potential is the first member ($l = 1$) of this subclass; for l even they have double-well structure, and the ϕ-four potential is the first member ($l = 2$) of this subclass.

Although the SG and ϕ-four potentials are the only members of this class which can be expressed as Taylor series in ϕ, the entire class of parent potentials enjoys (by construction) the very attractive feature that the spectra of small oscillations about the kink solutions are known exactly (Sec. III). This knowledge allows a rather complete construction of kink-gas phenomenologies for the low-temperature statistical mechanics[4] of the entire class, and the derivation of generalized susceptibilities[8,10] of the kinks to external perturbations. These topics will be discussed in subsequent papers.

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