Chapter 3

Collective-Coordinate Perturbation Theory

Having reviewed many of the perturbation theories used to study soliton dynamics, we develop a new method based on a canonical transformation developed by Tomboulis. This transformation identifies a "center of mass" coordinate $X(t)$ which is found to satisify a Newtonian equation of motion (to lowest order)

$$M_0 \ddot{X} = F(\dot{X}, t) \ ,$$

adding to the existing evidence [30, 31, 32, 37, 42, 44] which indicates that the kink does indeed behave like an extended Newtonian particle (for low velocities). The extended nature of this particle becomes apparent when the exact form of the force on the kink is examined. Before deriving this force, we quickly review some of the important facts about the small oscillations about nonlinear Klein-Gordon kinks.

3.1 Small Oscillations

In this section we briefly review the main features of solutions to the nonlinear Klein-Gordon class of field theories and the canonical transformation which forms the basis for our perturbation theory. The single-kink solutions to the wave equations along with small oscillations about these kinks will be described. The various quantities described in this section are collected in Table I for the sine-Gordon, ϕ^4, and double-quadratic potentials (this table corrects some errors in Table 3.1 of Ref. [75] and a similar error in Eq. 4.16b in Ref. [76]).

The general nonlinear Klein-Gordon Lagrangian we consider has the form

$$L = \int_{-\infty}^{\infty} dx \Big\{ \frac{1}{2}\Phi_t^2 - \frac{1}{2}\Phi_x^2 - U(\Phi) \Big\} \ , \tag{3.1.1}$$

Table 3.1: Various quantities for the ϕ^4, SG and DQ systems

where x and t are dimensionless space and time variables and $U(\phi)$ is the nonlinear potential with at least two degenerate minima. The nonlinear wave equation satisified by $\Phi(x,t)$ is

$$\Phi_{tt} - \Phi_{xx} + U'(\Phi) = 0 \ , \tag{3.1.2}$$

where the prime on $U(\Phi)$ denotes a derivative with respect to Φ. Static single-kink solutions, $\phi_c(x)$, of Eq. (3.1.2) may be obtained by direct integration with the boundary conditions

$$\frac{d\phi_c(x)}{dx}\Big|_{x=\pm\infty} = 0 \ , \tag{3.1.3}$$

The static kink (+) and antikink (-) solutions are given by

$$x = \pm\frac{1}{\sqrt{2}}\int\limits_{\phi_c(0)}^{\phi_c(x)} \frac{d\phi}{\sqrt{U(\phi)}} \ , \tag{3.1.4}$$

Moving solutions can be obtained by a Lorentz boost.

The equation governing the small oscillations about the static kink waveform is obtained by substituting

$$\Phi(x,t) = \phi_c(x) + \psi(x,t) \ , \tag{3.1.5}$$

into Eq. (3.1.2) and linearizing in ψ:

$$\psi_{tt} - \psi_{xx} + \psi U''[\phi_c(x)] = 0 \ . \tag{3.1.6}$$

Here $U''[\phi_c(x)]$denotes the second derivative of $U(\phi)$ with respect to ϕ evaluated for $\phi = \phi_c(x)$. Writing ψ as

$$\psi(x,t) = f(x)e^{i\omega t} \ , \tag{3.1.7}$$

leads to the following eigenvalue equation:

$$-f_{xx} + U''[\phi_c(x)]f = \omega^2 f \ . \tag{3.1.8}$$

Due to the localized nature of the kink waveform $\phi_c(x)$, the function $U''[\phi_c(x)]$ varies mainly in the region of the kink center (assumed to be at $x = 0$) and approaches a constant (taken to be unity) far from the kink center:

$$U''[\phi_c(x)] \xrightarrow[|x|\to\infty]{} 1 \ . \tag{3.1.9}$$

Moreover, the function $U''[\phi_c(x)]$ has a minimum at x=0 such that

$$U''[\phi_c(0)] < 0 \ . \tag{3.1.10}$$

From these properties, we see that there exists a close analogy between Eq. (3.1.8) and the Schrödinger equation for a "particle" moving in a one-dimensional "potential well", $U''[\phi_c(x)]$. The "bound state(s)" and "continuum" states for this potential are of fundamental importance for statistical mechanics phenomenologies [76], perturbation theories for kink dynamics [76, 37], and quantization procedures for kink states [77, 45, 47, 78, 79, 80, 81].

Since the Lagrangian (3.1.1) possesses translational invariance, the spectrum of the small oscillations about the single kink must contain a zero-frequency ($\omega = 0$) "translation" mode (Goldstone mode) which restores the translational invariance broken by the introduction of the kink. In addition to this translation mode there may be other discrete eigenvalues ("bound states") with frequencies between 0 and 1. These solutions, denoted by $f_{b,i}(x)$, correspond to "internal" oscillation modes in which the kink undergoes a harmonically varying shape change localized about the kink center. We denote these bound-state eigenfrequencies by $\omega_{b,1} \ldots \omega_{b,N}$ where N is the total number of bound states. The lowest of these is $\omega_{b,1} = 0$ since all other $\omega_{b,i}^2$ must be non-negative in order for the kink to be stable against small oscillations.

In addition to the bound states, there exist continuum states (continuous spectra) which are labelled by a wavevector k. These states have eigenvalues ω_k^2 given by

$$\omega_k^2 = 1 + k^2 \ . \tag{3.1.11}$$

which is precisely the dispersion relation for small oscillations in the absence of kinks.

The continuum states together with the bound-states form a complete set and satisfy the completeness relation,

$$\sum_{i=1}^{N} f_{b,i}^*(x) f_{b,i}(x') + \int_{-\infty}^{\infty} dk f_k^*(x) f_k(x') = \delta(x - x') \ . \tag{3.1.12}$$

and the following orthogonality relations:

$$\begin{aligned} \int_{-\infty}^{\infty} dx f_{b,n}(x) f_{b,m}(x) &= \delta_{m,n} \ , \\ \int_{-\infty}^{\infty} dx f_k^*(x) f_{k'}(x) &= \delta(k - k') \ , \\ \int_{-\infty}^{\infty} dx f_k(x) f_{b,n}(x) &= 0 \ . \end{aligned} \tag{3.1.13}$$

In the case in which $\phi_c'(x)$ is symmetric, $U''[\phi_c(x)]$ is also symmetric and therefore the continuum states may be chosen to have definite parity (if desired):

$$f_k(-x) = \pm f_{-k}(x) = \pm f_k^*(x) \ . \tag{3.1.14}$$

In addition, the following identity has been recently found [82]

$$\sum_{i=2}^{N} \frac{1}{\omega_{b,i}^2} f_{b,i}^*(x) \int_{-\infty}^{\infty} dy f_{b,i}(y)\phi_c''(y) + \int_{-\infty}^{\infty} dk \ \frac{1}{\omega_k^2} f_k^*(x) \int_{-\infty}^{\infty} dy f_k(y)\phi_c''(y) = \frac{1}{2} x\phi_c'(x) \ , \tag{3.1.15}$$

which can be readily proved by applying $-\partial^2/\partial x^2 + U''[\phi_c(x)]$ to both sides of Eq. (3.1.15).

Having illustrated the complete set of states associated with the linear operator given in Eq. (3.1.6), one might conclude that all of the relevant solutions of Eq. (3.1.6) have been found. This is not the case as has been pointed out in some recent work of Magyari and Thomas [83]. In searching for solutions of Eq. (3.1.6) we assumed a harmonic time dependence. However in general we should make the separation of variables ansatz

$$\psi = Q(t)f(x) \ . \tag{3.1.16}$$

Substituting this separation ansatz into Eq. (3.1.6) we obtain the following equation for $Q(t)$:

$$Q''(t) + \lambda Q(t) = 0 \ , \tag{3.1.17}$$

where λ is the separation constant. For $\lambda \neq 0$ we do indeed find a harmonic time dependence for $Q(t)$, however for $\lambda = 0$ there exists another solution which is linear in time

$$Q(t) = at + b \ . \tag{3.1.18}$$

Therefore we find that we have a degeneracy of the zero frequency eigenvalue. Just as the original zero frequency can be interpreted as having the effect of translating the kink in space, the second solution, termed the "defective degeneracy" [83], has the effect of inducing an infinitesimal velocity change in the kink. To see this we simply add the second solution to a stationary kink

$$\phi_c(x + \epsilon t) \approx \phi_c(x) + \epsilon t \phi_c'(x) \ . \tag{3.1.19}$$

The physical significance of this defective degeneracy can be further illustrated by considering the effects of an additional damping term $\eta\Phi_t$ to Eq. (3.1.2). Using the same separation ansatz we obtain the following equation for $Q(t)$

$$Q''(t) + \eta Q'(t) + \lambda Q(t) = 0 \ . \tag{3.1.20}$$

Again one can assume a harmonic time dependence

$$Q(t) = e^{-i\omega_{1,2}t} \ , \tag{3.1.21}$$

with $\omega_{1,2}$ given by

$$\omega_{1,2} = \pm\sqrt{\lambda - \frac{1}{4}\eta^2} - \frac{1}{2}i\eta \ . \tag{3.1.22}$$

Now the eigenvalue $\lambda = 0$ is no longer degenerate since in this case we obtain $\omega = 0$ (Goldstone mode) and $\omega = -i\eta$. Therefore with the addition of damping, the defective degeneracy of the $\lambda = 0$ eigenvalue is lifted by the occurence of the "relaxation mode"

$$Q(t) = e^{-\eta t} \ , \tag{3.1.23}$$

which describes the deceleration of an infinitesimally slowly moving kink to a shifted ($\phi_c'(x)$ still shifts the kink) static kink. This "degree of freedom" is actually accounted for by the kink velocity coordinate $\dot{X}(t)$ which is introduced in the next section.

3.2 The Collective Coordinate and the Canonical Transformation

As mentioned in Chapter 2, a transformation to a set of variables in which one can easily identify a kink coordinate would be of great utility when one describes the motion of a kink. Such a transformation has been found [45, 47] for *unperturbed* nonlinear Klein-Gordon field theories. This transformation decomposes the full field $\Phi(x,t)$ into a classical kink solution $\phi_c(x)$ whose center translates according to the dynamical variable $X(t)$ plus a "radiation" field $\chi(x,t)$

$$\Phi(x,t) = \phi_c(x - X(t)) + \chi(x - X(t), t) \tag{3.2.1}$$

The momentum conjugate to the field $\phi(x,t)$ is also decomposed into a soliton component plus a radiation field

$$\Pi_0(x,t) = \pi(x - X(t), t) - \frac{p + \int \pi\chi'}{M_0(1 + \xi/M_0)}\phi_c'(x - X(t)) \ , \tag{3.2.2}$$

where $M_0 \equiv \int \phi_c'\phi_c'$ and $\xi \equiv \int \chi'\phi_c'$. The prime denotes differentiation with respect to the argument and unless otherwise specified, all integrals denote one-dimensional integrals over x.

Having made the transformation

$$\{\Phi(x,t), \Pi_0(x,t)\} \rightarrow \{X(t), p(t), \chi(x,t), \pi(x,t)\} \ , \tag{3.2.3}$$

one notices that the number of degrees of freedom are not conserved, that is, on the left-hand side of Eq. (3.2.3) we have two full field degrees of freedom whereas on the right-hand side we have two full field degrees of freedom plus two discrete degrees of freedom. To remedy this situation, we must impose the following two constraints

$$\begin{aligned}\int_{-\infty}^{\infty} dx\ \chi(x,t)\phi_c'(x) &= 0\ ,\\ \int_{-\infty}^{\infty} dx\ \pi(x,t)\phi_c'(x) &= 0\ .\end{aligned} \tag{3.2.4}$$

The first of these constraints has the interpretation that the ψ field cannot have a term proportional to the translation mode, that is, the effect of the ψ field cannot cause the kink to translate. This is a very reasonable constraint since we have a dynamical variable whose only purpose is to translate the kink. The second of the constraints has a similar interpretation.

This transformation is not typical due to the presence of the constraints. To show that it is a canonical transformation, one must resort to the Dirac formalism for constrained systems [84]. Usually when one deals with constrained systems, the constraints cannot be used until all of the Poisson brackets have been taken, that is, the equalities in Eqs. (3.2.4) are "weak equalities". The Dirac formalism makes these equalities strong by modification of the brackets (Dirac brackets). For this particular transformation the brackets are

$$\Big\{\chi(x,t),\pi(y,t)\Big\} = \delta(x-y) - \frac{1}{M_0}\phi_c'(x)\phi_c'(y)\ , \tag{3.2.5}$$

$$\Big\{X,p\Big\} = 1\ , \tag{3.2.6}$$

with all remaining brackets vanishing. Using these brackets one can verify that the brackets in terms of the original variables satisfy the standard relations, that is

$$\Big\{\Phi(x,t),\Phi(y,t)\Big\} = \Big\{\Pi_0(x,t),\Pi_0(y,t)\Big\} = 0\ . \tag{3.2.7}$$

$$\Big\{\Phi(x,t),\Pi_0(y,t)\Big\} = \delta(x-y)\ , \tag{3.2.8}$$

In what follows, every time a bracket appears it is meant to indicate a Dirac bracket.

We can gain further insight into the transformation by examining the form which some of the energy-momentum tensor elements take in terms of the new variables. First we consider the Hamiltonian

$$\begin{aligned}H &= \int T_{00}\\ &= \int\left\{\frac{1}{2}\Pi_0^2 + \frac{1}{2}\Phi'^2 + U(\Phi)\right\}\ ,\\ &= M_0 + \frac{1}{2M_0}\frac{(p+\int \pi\chi')^2}{(1+\xi/M_0)^2} + \int \mathcal{H}_f\ ,\end{aligned} \tag{3.2.9}$$

where

$$\mathcal{H}_f = \frac{1}{2}\pi^2(x,t) + \frac{1}{2}\chi'^2(x,t) + V(\chi,\phi_c) \ , \tag{3.2.10}$$

$$V(\chi,\phi_c) = U(\phi_c+\chi) - \chi(x,t)U'(\phi_c) - U(\phi_c) \ , \tag{3.2.11}$$

where primes denote differentiation with respect to the argument and repeated use of

$$\phi_c'' = U'(\phi_c) \ , \tag{3.2.12}$$

and

$$\frac{1}{2}(\phi_c')^2 = U(\phi_c) \ , \tag{3.2.13}$$

has been made. Given the Hamiltonian in terms of the new variables, one can derive the equations of motion for the dynamical variables. In particular we find that the equation for X is given by

$$\dot{X} = \frac{p + \int \pi\chi'}{M_0(1+\xi/M_0)^2} \tag{3.2.14}$$

Using this equation of motion for X we can rewrite the Hamiltonian as

$$H = M_0 + \frac{1}{2}M_0(1+\xi/M_0)^2\dot{X}^2 + \int \mathcal{H}_f \ . \tag{3.2.15}$$

In this form we see that H almost decouples into a kink contribution and a phonon contribution. The term which prevents this decoupling can be understood to be a renormalization of the mass M_0. This coupling represents the interaction of the phonons back on the kink and is one the most interesting aspects of this method.

Next we turn our attention to the total momentum of the system

$$\begin{aligned}
P &\equiv \int T_{01} = -\int T^{01} \ , & (3.2.16)\\
&= \int \Pi_0(x,t)\Phi'(x,t) \ , & (3.2.17)\\
&= \int \pi(x-X,t)\Big[\phi_c'(x-X) + \chi'(x-X,t)\Big] & \\
&- \frac{p+\int\pi\chi'}{M_0(1+\xi/M_0)}\int\Big[\phi_c'(x-X)\chi'(x-X,t) + \phi_c'(x-X)\phi_c'(x-X)\Big] & \\
& & (3.2.18)\\
&= \int \pi\chi' - \frac{p+\int\pi\chi'}{M_0(1+\xi/M_0)}(M_0+\xi) & (3.2.19)\\
&= p \ . & (3.2.20)
\end{aligned}$$

From this we see that the variable p actually represents the total momentum of the system and not the kink momentum, even though it is conjugate to $X(t)$. It might appear that this would be rather inconvenient when one wishes to interpret the solutions to the equations of motion. This is not the case since in the following section we derive a second order equation for X. However it does present difficulties when we want to fix kink degrees of freedom in the Fokker-Planck method to be developed in Chapter 6. This problem can be avoided by using a transformation in which the momentum conjugate to the center of mass variable X is the kink momentum. The transformation for which p is the kink momentum is given by

$$\Phi(x,t) = \phi_c(x - X(t)) + \chi(x,t) \tag{3.2.21}$$

$$\Pi_0(x,t) = \pi(x,t) - \frac{p + \int \pi\chi'}{M_0(1+\xi/M_0)}\phi_c'(x - X(t)) \ , \tag{3.2.22}$$

with the constraints

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} dx \ \chi(x,t)\phi_c'(x-X) &= 0 \ , \\
\int\limits_{-\infty}^{\infty} dx \ \pi(x,t)\phi_c'(x-X) &= 0 \ .
\end{aligned} \tag{3.2.23}$$

To complete the transformation the Dirac brackets must be presented. The bracket of χ with π is the same as before but the brackets of the phonon variables χ and π with the momentum p become

$$\Big\{\chi(x,t), p\Big\} = 1 - \frac{\xi}{M_0}\phi_c'(x-X) \ , \tag{3.2.24}$$

$$= 1 - \mathcal{P}_{\phi_c}\chi'(x-X) \tag{3.2.25}$$

$$\Big\{\pi(x,t), p\Big\} = 1 - \frac{\phi_c'(x-X)}{M_0}\int\limits_{-\infty}^{\infty} dx \ \phi_c'(x-X)\pi'(x,t) \ , \tag{3.2.26}$$

$$= 1 - \mathcal{P}_{\phi_c}\pi'(x-X) \ , \tag{3.2.27}$$

with all other brackets zero. In the last step we have introduced the "translation-mode" projection operator $\mathcal{P}_{\phi_c}$ defined by

$$\mathcal{P}_{\phi_c}G(x,t) = \frac{\phi_c'(x)}{M_0}\int \phi_c'(x)G(x,t) \ . \tag{3.2.28}$$

This operator projects out that piece of any function which "overlaps" with the translation mode $\phi_c'(x)$.

In the actual calculation of the equations of motion one does not gain any advantage with either of the two brackets. The second transformation has the advantage that p is the kink momentum while the first, which is the one which will be implemented in the following chapters, has the virtue that the phonon field translates with the kink center of mass. This is especially useful when the interaction of the kink with the perturbation results in a permanent distortion of the kink waveform.

Finally we note that with the additional definition of a Lorentz boost generator

$$L = \int xT_{00} \ , \tag{3.2.29}$$

Tomboulis has shown [45] that the three operators H, P, and L form a Poincaré algebra,

$$\begin{aligned} \{H, P\} &= 0 \\ \{L, P\} &= H \\ \{L, H\} &= P \ , \end{aligned} \tag{3.2.30}$$

and therefore the unperturbed transformation preserves the Lorentz invariance evident in the Lagrangian.

3.3 The Perturbed System

The types of perturbations which we study have interaction Hamiltonians which may be written in the form

$$H_{int} = -\int_{-\infty}^{\infty} dx \ v(x,t) \ F[\Phi(x,t), \Phi_x(x,t)] \ , \tag{3.3.1}$$

where $v(x,t)$, assumed small in magnitude, denotes the space and time dependence of the perturbation and $F[\Phi(x,t), \Phi_x(x,t)]$ tells us how the perturbation couples to the field. The case in which the coupling function F is linear in the field and $v(x,t) = \delta(x - x_0)$ could be physically realized in terms of the pendulum chain if one of the pendula experienced a uniform external torque. Similarly for the pendulum chain, if $F = \Phi_x^2$ and $v(x,t) = \theta(x - x_0)$, the perturbation may be thought of as arising from an abrupt change in the spring constant of the chain. The general form of the perturbation (3.3.1) should allow many other types of perturbations to be examined, some of which will be presented in Chapter 5.

The aim of the perturbation theory is of course to study its influence on the kink. It should be kept in mind that even without the presence of a kink the perturbation influences the system. For example, if the perturbation is a torque on one of the pendula as mentioned above, the field in the vicinity of the applied torque will be modified as shown in Figure 3.1 This deformation will be present with or without the kink. If the kink scatters off of this perturbation,

Figure 3.1: Response of the sine-Gordon pendulum chain to a constant torque on a single pendulum

long before and after the scattering event the field in the region of the torque will be as shown in Figure 3.2 (neglecting any emitted phonons). The transformation described in the previous section could account for this feature through the χ field, however this is unattractive since then $\chi(x, \pm\infty)$ would be nonzero making the boundary conditions more difficult to deal with. Indeed, if this background response is not included from the outset, the field evolves in such a way as to "build up" this response and, in the process, the kink dynamics can appear [85] to be non-Newtonian. It is clear that one would like to take care of this deformation from the start and, in doing so, Newtonian dynamics is recovered. We accomplish this by introducing the "background" field $\psi_0(x,t)$ and modifying the canonical transformation to include it as follows

$$\begin{aligned} \Phi(x,t) &= \phi_c[x - X(t)] + \psi[x - X(t), t] + \psi_0(x,t) \ , && (3.3.2) \\ \Pi_0(x,t) &= \pi[x - X(t)] - \frac{p + \int \pi\psi'}{M_0(1+\xi/M_0)}\phi_c'[x - X(t)] - \dot{\psi}_0(x,t) \ , && (3.3.3) \end{aligned}$$

where M_0 and ξ are still defined by

$$M_0 = \int \phi_c'\phi_c' \ , \tag{3.3.4}$$

$$\xi = \int \psi'\phi_c' \ , \tag{3.3.5}$$

and the constraints are given by

$$\begin{aligned} \int dx\ \phi_c'(x)\psi(x,t) &= 0 & (3.3.6)\\ \int dx\ \phi_c'(x)\pi(x,t) &= 0\ . & (3.3.7) \end{aligned}$$

The ψ_0 term in Eq. (3.3.2) represents the response of the field to the perturbation *in the absence of a kink* and obeys the following equation:

$$[\partial_{tt} - \partial_{xx}]\psi_0(x,t) + \psi_0 U'(\psi_0) - F_{10}[\psi_0, \psi_0']v(x,t) + \frac{d}{dx}(v(x,t)F_{01}[\psi_0, \psi_0']) = 0\ , \tag{3.3.8}$$

with F_{ij} defined by

$$F_{ij} \equiv \frac{\partial^{i+j} F[\Phi, \Phi_x]}{\partial\Phi^i \partial\Phi_x^j}\ . \tag{3.3.9}$$

The field decomposition given in Eq. (3.3.2) is perhaps best illustrated with an example. Consider the "torqued pendulum" perturbation mentioned above with the kink scattering from $X = -\infty$ to $X = \infty$. In Figure 3.2 we show the field for times long before and after the scattering takes place. For $t = -\infty$ the kink has not yet interacted with the perturbation and therefore the field consists of the kink plus the background deformation. For large positive times the kink has scattered off of the perturbation and in the process emitted some phonons. These phonons are described by the ψ field while the ψ_0 field still accounts for the deformation in the region of the applied torque.

It has been shown [45] that Eqs. (3.3.2) and (3.3.3) specify a canonical transformation when no perturbation is present, that is for $\psi_0 = 0$ and $v(x,t) = 0$. That Eqs. (3.3.2) and (3.3.3) along with the constraints in Eqs. (3.3.6) and (3.3.7) still form a canonical transformation in the presence of a perturbation can be proved as follows. For no perturbation, Eqs. (3.3.2) and (3.3.3) are a point transformation of equations (3.2.1) and (3.2.2) and therefore the transformation is still canonical. The addition of a perturbing piece to the Hamiltonian has no effect since the canonical nature of the transformation depends only on the transformation equations and not on the Hamiltonian [86].

With the canonical transformation in hand, we may proceed to derive the equations of motion for the dynamical variables by using the Dirac-bracket formalism for constrained systems [84]. As in the previous section, the nonzero brackets for our system are

$$\begin{aligned} \{\psi(x,t), \pi(y,t)\} &= \delta(x-y) - \frac{\phi_c'(x)\phi_c'(y)}{M_0}\ , & (3.3.10)\\ \{X(t), p(t)\} &= 1 & (3.3.11) \end{aligned}$$

Figure 3.2: The various contributions to the field for the "torqued pendulum" perturbation. The solid line represents the kink contribution, the dashed line the background field ψ_0 and the dotted line the phonon portion ψ.

The bracket in Eq. (3.3.10) may be interpreted as a projection operator when it occurs under an integral sign which is always the case when the equations of motion are derived. For example, consider the following operation involving an arbitrary functional $G[\pi(y,t)]$ of the momentum field:

$$
\begin{aligned}
&\int dx\{\psi(x,t), G(\pi(y,t))\} \\
&= \int dx G'(\pi(y,t))\{\psi(x,t), \pi(y,t)\} && (3.3.12) \\
&= G'(\pi(x,t)) - \frac{\phi_c'(x)}{M_0}\int \phi_c'(x) G'(\pi(x,t)) && (3.3.13) \\
&= (1-\mathcal{P}_{\phi_c})G'(\pi(x,t)) \; . && (3.3.14)
\end{aligned}
$$

Given the brackets in Eqs. (3.3.10) and (3.3.11) , we may derive the equations of motions by taking the Dirac bracket of the dynamical variables with the Hamiltonian. Using the fact that the Hamiltonian in terms of the original variables is given by

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx \,\left[\Pi_0^2(x,t) + \Phi'^2(x,t) + U[\Phi]\right] + H_{int} \; , \tag{3.3.15}$$

we write the Hamiltonian in terms of the new variables as

$$H = H_0 + H_{\psi_0} + H_{int} \; , \tag{3.3.16}$$

where

$$H_0 = M_0 + \frac{1}{2M_0}\frac{(p+\int \pi\psi')^2}{(1+\xi/M_0)^2} + \int H_f \; , \tag{3.3.17}$$

$$H_f(x,t) = \frac{1}{2}\pi^2(x,t) + \frac{1}{2}\psi'^2(x,t) + V(\psi,\phi_c) \; , \tag{3.3.18}$$

and

$$V(\psi,\phi_c) = U(\phi_c+\psi) - \psi(x,t)U'(\phi_c) - U(\phi_c) \; . \tag{3.3.19}$$

H_{int} is given in Eq. (3.3.1) and H_{ψ_0} is defined by

$$
\begin{aligned}
H_{\psi_0} &= -\pi[x-X(t),t]\dot{\psi}_0'(x,t) + \frac{p+\int\pi\psi'}{M_0(1+\xi/M_0)}\phi_c'[x-X(t)]\dot{\psi}_0(x,t) \\
&+ \psi'[x-X(t),t]\psi_0'(x,t) + \phi_c'[x-X(t)]\psi_0'(x,t) + \Delta U \; , && (3.3.20)
\end{aligned}
$$

with

$$\Delta U = U[\phi(x) + \psi(x,t) + \psi_0(x+X(t),t)] - U[\phi(x)+\psi(x,t)] \; . \tag{3.3.21}$$

The calculation of the equations of motion for X, p, ψ, π is straightforward but tedious and is therefore relegated to Appendix A.

Since we are most interested in the kink center of mass motion, it is useful to derive a second order equation for the kink center of mass variable $X(t)$. Using Eq. (A.23) from Appendix A we have

$$\begin{aligned} M_0\ddot{X} &= \frac{1}{(1+\xi/M_0)}\Big\{-\int v(x,t)\Big[\phi_c'(x-X)F_{10}[\Phi,\Phi_x]+\phi_c''(x-X)F_{01}[\Phi,\Phi_x]\Big] \\ &+ \int\Big[\ddot{\psi}_0(x,t)-\psi_0''(x,t)\Big]\phi_c'(x-X)+\int\phi_c'(x-X)U'[\Phi(x,t)] \\ &+ (1+\dot{X}^2)\int\psi'\phi_c''-2\dot{X}\int\pi'\phi_c'+2\dot{X}\int\phi_c'(x)\dot{\psi}_0'(x+X,t)\Big\}\ . \end{aligned} \tag{3.3.22}$$

where Φ is understood to mean $\Phi(x,t)$. Since we have not yet made any approximations, Eq. (3.3.22) is *exact* and states that the kink center of mass variable $X(t)$ obeys Newton's law. The "force" that the kink experiences, that is the right-hand side of Eq. (3.3.22), has several interesting properties. First, it includes terms which depend on the radiation field $\psi(x,t)$ and therefore the equations that must be solved are really a set of integro-differential equations which are most easily solved perturbatively. Physically, the presence of ψ in the kink equations means that any phonons produced by the propagation of the kink in the perturbed system in turn affect the kink's motion. The second interesting feature of the "force" on the kink is that one of the terms is proportional to the square of the kink velocity, that is, there is a "dissipative" term in the center of mass equation of motion. Because we started with a Hamiltonian system, this "dissipative" term cannot represent a real loss of energy. Rather, this term represents a transfer of energy between the kink center of mass motion and the radiation field.

One is tempted to interpret the term which is linear in $\dot{X}$ as also representing a transfer of energy. However, further examination of Eq. (3.3.22) indicates that this term is actually part of the inertia of the kink. To see this we make use of Eq. (A.13) of Appendix A to replace the $\pi'(x,t)$ term in Eq. (3.3.22) by an equivalent expression in terms of the ψ and ψ_0 fields:

$$\int\pi'\phi_c'=\int\dot{\psi}'\phi_c'-\dot{X}\int\psi''\phi_c'+\int\dot{\psi}_0'(x+X,t)\phi_c'(x)\ . \tag{3.3.23}$$

Substitution of this expression in to Eq. (3.3.22) yields

$$\begin{aligned} M_0\ddot{X} &= \frac{1}{(1+\xi/M_0)}\Big\{-\int v(x,t)\Big[\phi_c'(x-X)F_{10}[\Phi,\Phi_x]+\phi_c''(x-X)F_{01}[\Phi,\Phi_x]\Big] \\ &+ \int\Big[\ddot{\psi}_0(x,t)-\psi_0''(x,t)\Big]\phi_c'(x-X)+\int\phi_c'(x-X)U'[\Phi(x,t)] \\ &+ (1-\dot{X}^2)\int\psi'\phi_c''-2\dot{X}\int\dot{\psi}'\phi_c'\Big\}\ . \end{aligned} \tag{3.3.24}$$

Next we move the term linear in $\dot{X}$ to the left-hand side of the equation and multiply by $1 + \xi/M_0$ which allows us to write

$$\begin{aligned}
&\frac{d}{dt}\Big[M_0(1+\xi/M_0)^2\dot{X}\Big] \\
&= (1+\xi/M_0)\Big\{-\int v(x,t)\Big[\phi_c'(x-X)F_{10}[\Phi,\Phi_x] + \phi_c''(x-X)F_{01}[\Phi,\Phi_x]\Big] \\
&+ \int\Big[\ddot{\psi}_0(x,t) - \psi_0''(x,t)\Big]\phi_c'(x-X) + \int \phi_c'(x-X)U'[\Phi(x,t)] \\
&+ (1-\dot{X}^2)\int \psi'\phi_c''\Big\} \ ,
\end{aligned} \tag{3.3.25}$$

where we have made use of the fact that

$$\dot{\xi} = \int \dot{\psi}'\phi_c' \ . \tag{3.3.26}$$

Equation (3.3.25) is nothing more that Newton's law for a particle with time-dependent mass

$$M = M_0(1+\xi/M_0)^2 \ . \tag{3.3.27}$$

Therefore we see that one of the effects of the phonon field is to renormalize the mass of the kink. This feature of the phonon field has already been noted in the quantized theories for the unperturbed system [47]. The interpretation of the left-hand side of Eq. (3.3.25) as being the time derivative of the kink momentum is verified by examining the equation for $\dot{X}$ derived in Appendix A. Rewriting Eq. (A.11) we have

$$M_0(1+\xi/M_0)^2\dot{X} = p + \int \pi\psi' + (1+\xi/M_0)A(X,t) \ , \tag{3.3.28}$$

which states that the kink momentum equals the total momentum of the system p minus the momentum of the phonon field ($-\int \pi\psi'$) minus a momentum term due to the background.

Equation (3.3.22) was derived by using the fact that Eqs. (3.3.2) and (3.3.3) plus the constraints form a canonical transformation and therefore the Dirac bracket of the dynamical variables with the Hamiltonian yields the equations of motion. An alternate method is available which uses the fact that in Eqs. (3.3.2) and (3.3.3) we have a transformation in which the old coordinates are expressible in terms of the new coordinates and therefore, one can derive the equations of motion simply by taking the appropriate derivatives of Eqs. (3.3.2) and subsituting them into Eq. (3.1.2). In fact, we can generalize Eq. (3.1.2) by including a phenomenological damping term of the form

$$\epsilon\dot{\Phi}(x,t) \ .$$

Unlike the "damping" terms in Eq. (3.3.22), this term is truly dissipative and may be envisioned as arising from coupling the system to a heat bath.

Although substitution of Eq. (3.3.2) into Eq. (3.1.2) (see Appendix B) yields the correct equations of motion more quickly and with less work than using the canonical formalism, this in no way means that we can abandon the canonical transformation. A major reason for this is that the canonical structure allows us to use the standard prescription of promoting canonical variables to operators and the Poisson bracket to commutators when we wish to quantize the system [45, 47]. In addition, when one works with phase space integrals as is the case in Fokker-Planck (see Chapter 6) or Boltzmann approaches, having a canonical transformation preserves the phase space volume element (for our transformation which involves constraints, the Jacobian is actually the product of the delta functions whose arguments are exactly the constraints in Eqs. (3.2.4) [47]).

3.4 The Perturbation Expansion

We now turn our attention to the task of obtaining an approximate solution of Eq. (3.3.24) as a perturbation series. We assume that the perturbation $v(x,t)$ is proportional to some small parameter which we denote by λ. Since we are mainly interested in the motion of the kink center of mass, we begin by expanding Eq. (3.3.2). For these purposes, we assume that $v(x,t)$, $\psi(x,t)$ and $\psi_0(x,t)$ are all of order λ. To obtain the expansion of Eq. (3.3.24) through order λ^2, we make use of the following Taylor series

$$\begin{aligned} F_{10}[\Phi(x+X,t),\Phi_x(x+X,t)] = \frac{\partial F(\phi_c,\phi_c')}{\partial\phi_c} &+ \chi(x,t)\frac{\partial^2 F(\phi_c,\phi_c')}{\partial^2\phi_c^2} \\ &+ \chi'(x,t)\frac{\partial^2 F(\phi_c,\phi_c')}{\partial\phi_c'\partial\phi_c} \end{aligned} \tag{3.4.1}$$

$$\begin{aligned} F_{01}[\Phi(x+X,t),\Phi_x(x+X,t)] = \frac{\partial F(\phi_c,\phi_c')}{\partial\phi_c'} &+ \chi'(x,t)\frac{\partial^2 F(\phi_c,\phi_c')}{\partial^2\phi_c'^2} \\ &+ \chi(x,t)\frac{\partial^2 F(\phi_c,\phi_c')}{\partial\phi_c'\partial\phi_c} \end{aligned} \tag{3.4.2}$$

where for notational convenience we have introduced

$$\chi(x,t) = \psi(x,t) + \psi_0(x+X,t) \ . \tag{3.4.3}$$

Substituting Eqs. (3.4.1) and (3.4.2) into Eq. (3.3.24) we have, after collecting terms

$$M_0\ddot{X} \;=\; \frac{1}{(1+\xi/M_0)}\Big\{-\int v(x+X,t)\Big[\frac{d}{dx}F(\phi_c,\phi_c')$$

$$
\begin{aligned}
&+ \quad \chi(x,t)\frac{d}{dx}\frac{\partial F(\phi_c,\phi_c')}{\partial \phi_c} + \chi'(x,t)\frac{d}{dx}\frac{\partial F(\phi_c,\phi_c')}{\partial \phi_c'}\Big] \\
&+ \quad \int (\ddot{\psi}_0(x,t) - \psi_0''(x,t))\phi_c'(x-X)) + \int \phi_c'(x-X)U'[\Phi(x,t)] \\
&+ \quad (1-\dot{X}^2)\int \psi'\phi_c'' - 2\dot{X}\int \dot{\psi}'\phi_c' \Big\} \ . \qquad (3.4.4)
\end{aligned}
$$

Next we write the following Taylor series for $U'[\Phi(x+X,t)]$

$$
U'[\Phi(x+X,t)] = U'[\phi_c(x,t)] + \frac{1}{2}\chi(x,t)U''[\phi_c(x,t)] + \chi^2(x,t)U'''[\phi_c(x,t)] \ , \qquad (3.4.5)
$$

where we have used our freedom to choose a normalization such that the following are true

$$
U'[\phi_0] = 0 \quad , \quad U''[\phi_0] = 1 \ , \qquad (3.4.6)
$$

where the potential U has its minimum at ϕ_0. Combining all these terms we have

$$
\begin{aligned}
M_0\ddot{X} &= -\Big(1 - \frac{\xi}{M_0}\Big)\left[\frac{\partial V(X,t)}{\partial X} + 2\dot{X}\int \dot{\psi}'\phi_c' + \dot{X}^2\int \psi\phi_c''\right] \\
&- \quad \int v(x+X,t)\Big[\chi(x,t)\frac{d}{dx}\frac{\partial F(\phi_c,\phi_c')}{\partial \phi_c} + \chi'(x,t)\frac{d}{dx}\frac{\partial F(\phi_c,\phi_c')}{\partial \phi_c'}\Big] \\
&+ \quad \frac{1}{2}\int \chi^2(x,t)U'''[\phi_c(x)]\phi_c'(x) \ , \qquad (3.4.7)
\end{aligned}
$$

where the effective potential $V(x,t)$ is defined by

$$
V(X,t) = -\int \Big[v(x+X,t)F(\phi_c,\phi_c') - \ddot{\psi}_0(x,t)\phi_c'(x-X)\Big] \ . \qquad (3.4.8)
$$

Equation (3.4.7) is valid through second order in λ. Keeping only the first order terms in λ we have

$$
M_0\ddot{X} = -\frac{\partial V(X,t)}{\partial X} - 2\dot{X}\int \dot{\psi}'\phi_c' - \dot{X}^2\int \psi\phi_c'' \ . \qquad (3.4.9)
$$

Although in principle one must know the phonon field ψ to lowest order in λ before Eq. (3.4.9) can be solved, in practice the terms which involve the phonon field are often small relative to the gradiant of the potential. In order to estimate the magnitude of the phonon field, one must solve the first-order PDE for ψ which is derived below. This in turn requires knowledge of the first-order kink motion. Therefore in principle one must solve a set of coupled equations self-consistently. To make progress without solving the coupled equations, one *assumes* that the ψ field is small. This allows one to solve for the first-order kink motion

$$
M_0\ddot{X} = -\frac{\partial V(X,t)}{\partial X} \ . \qquad (3.4.10)
$$

Given $X(t)$ to lowest order one proceeds to solve the equation for the phonon field (see below). Now one is in a position to check if the gradient term does indeed dominant in Eq. (3.4.9). If this is the case, one may continue to higher order. When the gradient and "phonon" terms are of the same order of magnitude, one must solve the coupled equations self-consistently. This will be the case when the size of the perturbation, that is λ, is much smaller than the initial velocity of the kink. An alternate method involves making a perturbation expansion in both the strength of the perturbation and the initial kink velocity.

A second-order PDE for ψ, valid to first order in the perturbation expansion is derived in Appendix A and is given by

$$\begin{aligned}\ddot{\psi}(x,t)-\psi''(x,t) &+ \psi(x,t)U''(\phi_c)=(1-\mathcal{P}_{\phi_c})\Big\{\big[1-U''(\phi_c)\big]\psi_0(x+X,t)\\ &+ v(x+X,t)\big[F_{10}[\phi_c,\phi_c']-F_{10}[0,0]\big]\\ &- \frac{d}{dx}\Big[v(x+X,t)\big(F_{01}[\phi_c,\phi_c']-F_{01}[0,0]\big)\Big]\Big\}\ . \end{aligned} \tag{3.4.11}$$

Since the left-hand side of Eq. (3.4.11) is exactly the small oscillations operator of Eq. (3.1.6), we can write the a solution for $\psi(x,t)$ is terms of a Green function

$$\psi(x,t)=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dt'G(x,x',t-t')I(x',t')\ , \tag{3.4.12}$$

where

$$\begin{aligned}G(x,x',\tau) &= \sum_{i=1}^{N} f_{b,i}^*(x)f_{b,i}(x')\int_{-\infty}^{\infty}\frac{d\omega e^{i\omega\tau}}{2\pi(\omega_{b,i}^2-\omega^2)}\\ &+ \int_{-\infty}^{\infty} f_k^*(x)f_k(x')\int_{-\infty}^{\infty}\frac{d\omega e^{i\omega\tau}}{2\pi(\omega_k^2-\omega^2)}\ ,\end{aligned} \tag{3.4.13}$$

with $\tau=t-t'$ and $I(x,t)$ is the right-hand side of Eq. (3.4.11). The Green function in Eq. (3.4.13) contains all of the bound states including the translation mode. At first this seems to contradict the constraint condition in Eq. (3.3.6) because including the translation mode $f_{b,1}(x)$ in the Green function means that $\psi(x,t)$ could have a portion proportional to the translation mode, namely

$$\psi(x,t)=-f_{b,1}^*(x)\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dt'f_{b,1}(x')I(x',t')\int_{-\infty}^{\infty}\frac{d\omega e^{i\omega\tau}}{2\pi(\omega_k^2-\omega^2)}\ . \tag{3.4.14}$$

However this is not the case because the expression $I(x',t')$ is manifestly orthogonal to the translation mode and therefore we will find no coefficient of the translation mode. Analytic expressions for the Green function defined in Eq. (3.4.13) in terms of modified Lommel functions of two variables are derived in Chapter 4 for the sine-Gordon, Phi-4 and Double Quadratic potentials [87].

Although one can in principle calculate the ψ field using the Green function, one finds in many cases that it is more cost effective to solve the partial differential equation (3.4.11) directly. One might question the utility of the present perturbation theory if one must, in the end, numerically solve a PDE for ψ when numerical integration of Eq. (3.1.2) solves the entire problem. From a purely computational point of view we could argue that Eq. (3.1.2), unlike Eq. (3.4.11), is a strongly nonlinear PDE and therefore, although there are subroutine packages available which can handle these equations [88], they are often quite costly. Furthermore, although Eq. (3.4.1) may be cumbersome to use in practice, it can be used to deduce general features of the ψ field. For example, consider the situation in which a kink is incident upon a time-independent localized perturbation and scatters to $X = \pm\infty$ (see §5.3). In this case, ψ_0 is time independent and the only time dependence which enters the terms $\psi_0(x + X, t)$ and $v(x + X, t)$ occurring in $I(x, t)$ is through $X(t)$. If the kink scatters to $X = \pm\infty$, for large times $X(t)$ will vary as $V_0 t$ to first order. Since $v(x)$ is localized, both v and ψ_0 will be localized in their first argument. Therefore for large values of t, $\psi_0(x' + X(t'))$ and $v(x' + X(t'))$ will contribute to the integral in Eq. (3.4.1) only for small values of t'. Using the fact that the asymptotic time dependence for the Green function is (see §4.2)

$$G(x, x', t - t') \approx \frac{1}{\sqrt{t - t'}} \ , \tag{3.4.15}$$

one can write

$$\begin{aligned} \psi(x, t) &\approx \int_{-\infty}^{\infty} dx' \int_{t}^{\infty} dt' \frac{I(x', t')}{\sqrt{t - t'}} \ , & (3.4.16) \\ &\approx \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} dx' \int_{t}^{\infty} dt' \frac{I(x', t')}{\sqrt{1 - t'/t}} \ . & (3.4.17) \end{aligned}$$

Therefore for fixed x, the ψ field tends to zero for large times, that is to say, there are no long-lived extended phonons generated to this order in the perturbation theory. Similar results have been obtained by Wada and Schriefer [67] and Ogata and Wada [68] when they consider the scattering of a phonon packet (analogous to our ψ_0 field) with a kink. They find that the kink only undergoes a phase shift to lowest order and that no new phonons are emitted. The major difference between their work and ours is that we have a force which maintains the "phonon" packet's shape, however it appears that the effect of this force makes itself felt only in higher order.

Although this result is itself interesting, it has larger implications for the second-order kink motion. Consider the time-dependent second-order terms on the right hand side of Eq. (3.4.7). All of these terms involve either $\psi(x,t)$, which goes to zero as $t \to \infty$, or $\psi_0(x + X(t), t)$. For the scattering situation considered here, $X(t) \to \infty$ as $t \to \infty$; therefore due to the assumed localized nature of the perturbation, $\psi_0(x + X(t), t)$ also goes to zero for large times. Since we already know that the effective potential is zero for large X, the force on the kink for large times is zero. Therefore after the kink has interacted with the perturbation it travels at constant velocity. If some energy has been given to the phonon field this velocity should be less than the initial kink velocity.

The question of the final kink velocity may be attacked more generally by obtaining an approximate first integral of Eq. (3.4.7). To this end we note that given the background field ψ_0, the phonon field ψ, and the first order solution to the kink center of mass motion (i.e. given $X^{(1)}(t)$), the right hand side of Eq. (3.4.7), excluding the $\dot{X}$ term, can be written as a time dependent force denoted by $F(t)$. Also noting that the coefficient of the $\dot{X}$ term is precisely $2\dot{\xi}$, we write

$$(M_0 + \xi(t))\ddot{X} + 2\dot{\xi}\dot{X} = F(t) \ , \tag{3.4.18}$$

where we have also assumed that the $\dot{X}^2$ term is negligible. Multiplying by the integrating factor $(1 + \xi/M_0)$ Eq. (3.4.18) may be rewritten as

$$\frac{d}{dt}\Big[M_0(1 + \xi/M_0)^2\dot{X}\Big] = (1 + \xi/M_0)F(t) \ . \tag{3.4.19}$$

Integrating this equation from $t = -\infty$ to $t = \infty$ we obtain

$$\Big(1 + \frac{\xi(\infty)}{M_0}\Big)\dot{X}(\infty) - \Big(1 + \frac{\xi(-\infty)}{M_0}\Big)\dot{X}(-\infty) = \frac{1}{M_0}\int\limits_{-\infty}^{\infty} dt'\Big(1 + \frac{\xi(t')}{M_0}\Big)F(t') \ . \tag{3.4.20}$$

For the special case in which the perturbation is localized, $\xi(\pm\infty) = 0$ which allows us to reduce Eq. (3.4.20) to

$$\dot{X}(\infty) - \dot{X}(-\infty) = \frac{1}{M_0}\int\limits_{-\infty}^{\infty} dt'\Big(1 + \frac{\xi(t')}{M_0}\Big)F(t') \ . \tag{3.4.21}$$

These forms allow one to deduce the final kink velocity by performing one numerical integral (this is a very recent result and has not yet been implemented). The expression for the final velocity given in Eq. (3.4.20) should prove to be a good check on the numerical integation of Eq. (3.4.7).

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