Chapter 4

Nonlinear Klein-Gordon Green Functions

We conclude the formal derivation of the perturbation theory by calculating the Green functions needed to compute the phonon field ψ for the sine-Gordon, ϕ^4 and double quadratic potentials. Recall from section 3.4 that ψ may be expressed as

$$\psi(x,t) = \int\limits_{-\infty}^{\infty} dx' \int\limits_{-\infty}^{\infty} dt' G(x', x', t-t') I(x', t') \ ,$$

where $I(x', t')$ is an inhomogeneous term which depends on the perturbation. Although this expression for ψ is easy enough to write down, one must ask whether or not it is useful in practice. As mentioned in section 3.4, for the perturbations considered so far we have found that performing the integrals in Eq. (3.4.12) requires too much computation time (I estimate that to compute ψ to 3 significant digits for 1000 values of x and t would require roughly 1 hour of Cray 1 time). This computation requires a lot of time because the Green function oscillates rapidly in t' over the range of times t' for which the inhomogeneous term $I(x', t')$ is appreciable. When one encounters this type of behavior one immediately considers transforming to Fourier space where the Green functions would decay rapidly. This does not help in our case because we have imposed retarded boundary conditions on the Green functions which evidence themselves by the appearance of the step function prefactor $\theta(t - t')$. Next one considers the use of the Laplace transform. When one sees the rather complex analytic form the Green functions take it appears at first that this approach is not possible. It is indeed remarkable that we can obtain analytic forms for the Laplace transform of the Green functions (see section 4.3); however, one is then faced with the nontrivial task of numerically evaluating the Bromwich integral. Although these methods have not yet proved to be useful, it is quite possible that for special perturbations they could lead to analytic expressions for the phonon field.

In the following we derive the Green functions for the sine-Gordon, ϕ^4 and double quadratic nonlinear potentials. One might ask whether other nonlinear potentials could be examined with similar techniques. Since the sine-Gordon and ϕ^4 potentials are the first two of an infinite sequence of nonlinear potentials [49] it is conceivable that this sequence of potentials would be tractable. However, since the phonon waveforms are known analytically [49], we can see that the amount of work needed for each successive potential in the sequence increases linearly, so that one would need to develop a method which applied to the general potential. In addition, it might be desirable to apply different boundary conditions such as periodic boundary conditions on the finite line. However for now we content ourselves with the retarded conditions as applied to the potentials mentioned above.

4.1 Analytic Evaluation of the Green functions

For the set $\{f_{b,i}(x), f_k(x)\}$ of solutions satisfying the "phonon" equation (3.1.8), we define the full Green function as:

$$\begin{aligned} G(x,x',\tau) &= \sum_{bound\ states} f^*_{b,i}(x) f_{b,i}(x') \int_{-\infty}^{\infty} \frac{d\omega e^{i\omega\tau}}{2\pi(\omega_i^2-\omega^2)} \\ &+ \int_{-\infty}^{\infty} dk\ f^*_k(x) f_k(x') \int_{-\infty}^{\infty} \frac{d\omega e^{i\omega\tau}}{2\pi(\omega_k^2-\omega^2)} , \end{aligned} \tag{4.1.1}$$

where $\tau \equiv t-t'$. Using the completeness relation (3.1.12), and the fact that the set $\{f_{b,i}(x), f_k(x)\}$ satisify equation (3.1.8), one can show that the full Green function satisfies the usual equation:

$$\{\partial_{tt} - \partial_{xx} + U''[\phi_k(x)]\} G(x,x',\tau) = \delta(x-x')\delta(\tau) . \tag{4.1.2}$$

Once a set of boundary conditions is chosen the ω integral in (4.1.1) may be evaluated without choosing a particular set of $\{f_{b,i}(x), f_k(x)\}$. In this paper we choose retarded boundary conditions obtained by moving both of the poles in the ω integral above the real ω axis. Carrying out the ω integral yields:

$$G(x,x',\tau) = G_b(x,x',\tau) + G_p(x,x',\tau) , \tag{4.1.3}$$

where $G_b(x,x',\tau)$ and $G_p(x,x',\tau)$ are the bound state and phonon contributions given by:

$$G_b(x,x',\tau) = \theta(\tau)\left\{\tau f^*_{b,1}(x) f_{b,1}(x') + \sum_{i=2}^{N} f^*_{b,i}(x) f_{b,i}(x') \frac{\sin(\omega_i\tau)}{\omega_i}\right\} , \tag{4.1.4}$$

$$G_p(x,x',\tau) = \theta(\tau) \int_{-\infty}^{\infty} dk f^*_k(x) f_k(x') \frac{\sin(\omega_k\tau)}{\omega_k} , \tag{4.1.5}$$

with N the number of bound states [if N=1 the second term is omitted from Eq. (4.1.4)] and $\theta(\tau)$ is the Heaviside step function,

$$\theta(\tau) = \begin{cases} 0, & -\infty < \tau < 0 \\ 1 & 0 \leq \tau < \infty \end{cases} \tag{4.1.6}$$

In order to obtain explicit forms for these contributions to the Green function, one must insert the appropriate set of linearized solutions into Eqs. (4.1.4) and (4.1.5). As examples, we evaluate the phonon contribution for the SG, ϕ^4 and DQ potentials.

4.1.1 The sine-Gordon Potential

Since the bound state contribution (4.1.4) is already expressed in terms of known functions, we turn to the evaluation of the phonon contribution given in Eq. (4.1.5). Inserting the functions $f(x)$ from the SG column of Table 3.1 into Eq. (4.1.5) we have, after collecting common terms,

$$G_p^{SG}(x, x', \tau) = \theta(\tau)\{I_1 + \beta_2 I_2 + \beta_3 sgn(z) I_3\} \ , \tag{4.1.7}$$

where

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_0^\infty \frac{dk}{\sqrt{1+k^2}} \cos(|z|k)\sin(\tau\sqrt{1+k^2}) \ , \\ I_2 &= \frac{1}{\pi}\int_0^\infty \frac{dk}{(1+k^2)^{\frac{3}{2}}} \cos(|z|k)\sin(\tau\sqrt{1+k^2}) \ , \\ I_3 &= \frac{1}{\pi}\int_0^\infty \frac{dk}{(1+k^2)^{\frac{3}{2}}} k\sin(|z|k)\sin(\tau\sqrt{1+k^2}) \ , \end{aligned} \tag{4.1.8}$$

with the definitions

$$\tau \equiv t - t' \quad , z \equiv x - x', \quad \beta_2 \equiv \tanh(x)\tanh(x') - 1, \quad \beta_3 \equiv \tanh(x') - \tanh(x) \ . \tag{4.1.9}$$

Since I_2 is uniformly convergent for all $|z|$ and τ, we may differentiate with respect to $|z|$ to obtain

$$I_3 = -\frac{dI_2}{d|z|} \ . \tag{4.1.10}$$

Therefore only $I_1$and I_2 need to be evaluated. These integrals may be evaluated by considering the integral $I(\mu)$ given by

$$\begin{aligned} I(\mu) &= \frac{1}{\pi}\int_0^\infty \frac{dk}{\sqrt{\mu^2+k^2}} \cos(|z|k)\sin(\tau\sqrt{\mu^2+k^2}) \ , & (4.1.11) \\ &= \frac{\theta(\tau-|z|)}{2} J_0(\mu\sqrt{\tau^2-z^2}) \ , & (4.1.12) \end{aligned}$$

where the integral is found in the tables [89]. The special case $I(1)$, is precisely the integral I_1. Since the derivative of the integrand of Eq. (4.1.11) is a continuous function of both μ and k we may differentiate $I(\mu)$ with respect to μ to obtain

$$I_2 = \lim_{\mu\to 1}\left\{-\frac{dI(\mu)}{d\mu} + \frac{\tau}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\mu^2+k^2}\cos(|z|k)\cos(\tau\sqrt{\mu^2+k^2})\right\}, \quad (4.1.13)$$

$$= \frac{\theta(\tau-|z|)}{2}\sqrt{\tau^2-z^2}J_1(\sqrt{\tau^2-z^2}) + \frac{\tau}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{1+k^2}\cos(|z|k)\cos(\tau\sqrt{1+k^2})\,. \quad (4.1.14)$$

In the integral remaining in (4.1.14) we substitute $k = \sinh(u)$, which gives us

$$\frac{\tau}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{1+k^2}\cos(|z|k)\cos(\tau\sqrt{1+k^2}) \quad (4.1.15)$$

$$= \frac{\tau}{2\pi}\int_{-\infty}^{\infty}\frac{du}{\cosh(u)}\cos[|z|\sinh(u)]\cos[\cosh\tau(u)]\,, \quad (4.1.16)$$

$$= \frac{\tau}{4\pi}\int_{-\infty}^{\infty}\frac{du}{\cosh(u)}\Big\{\cos[|z|\sinh(u)+\tau\cosh(u)] + \cos[\tau\cosh(u)-|z|\sinh(u)]\Big\}\,, \quad (4.1.17)$$

$$= \frac{\tau}{2\pi}\int_{-\infty}^{\infty}\frac{due^u}{e^{2u}+1}\Big\{\cos[ae^u+be^{-u}]+\cos[ae^{-u}+be^u]\Big\}\,, \quad (4.1.18)$$

$$= \frac{\tau}{2\pi}\int_{0}^{\infty}\frac{dt}{t^2+1}\Big\{\cos[at+\frac{b}{t}]+\cos[\frac{a}{t}+bt]\Big\}\,, \quad (4.1.19)$$

$$= \frac{\tau}{\pi}\int_{0}^{\infty}\frac{dt}{t^2+1}\cos[at+\frac{b}{t}]\,, \quad (4.1.20)$$

where in passing from (4.1.19) to (4.1.20) we have let $t \to 1/t$ in the second cosine term and have defined

$$a \equiv \frac{\tau+|z|}{2}\,, \quad (4.1.21)$$

$$b \equiv \frac{\tau-|z|}{2}\,. \quad (4.1.22)$$

For $b < 0$ the integral in (4.1.20) is found in the tables [90] to be

$$\frac{1}{\pi}\int_{0}^{\infty}\frac{dt}{t^2+1}\cos\Big[at-\frac{|b|}{t}\Big] = \frac{1}{2}e^{(a-b)}. \quad (4.1.23)$$

For $b > 0$ the integral in Eq. (4.1.20) may be expressed in terms of "modified" Lommel functions of two variables [91]. The "modified" functions, namely Lommel functions in which the first argument is pure imaginary, have not been found in the literature. Hence we introduce the notation $\Lambda_n(w,s)$ and $\Xi_n(w,s)$ for the modified functions and give their series representations in terms of Bessel functions:

$$\Lambda_n(w,s) \equiv i^{-n}U_n(iw,s) = \sum_{m=0}^{\infty}\Big(\frac{w}{s}\Big)^{2m+n} J_{2m+n}(s)\ , \tag{4.1.24}$$

$$\Xi_n(w,s) \equiv i^{-n}V_n(iw,s) = \sum_{m=0}^{\infty}\Big(\frac{w}{s}\Big)^{-2m-n} J_{-2m-n}(s)\ , \tag{4.1.25}$$

With these definitions, we write for $b > 0$

$$\frac{1}{\pi}\int_0^{\infty}\frac{dt}{t^2+1}\cos[at+\frac{|b|}{t}] = \frac{1}{2}e^{-(a-b)} - \Lambda_1(w,s)\ , \tag{4.1.26}$$

where

$$s \equiv \sqrt{\tau^2 - z^2}\ , \tag{4.1.27}$$

$$w \equiv \tau - |z|\ . \tag{4.1.28}$$

Combining (4.1.23) and (4.1.26) we have for I_2 :

$$I_2 = \frac{1}{2}\tau e^{-|z|} + \theta(\tau-|z|)\Big\{\frac{sJ_1(s)}{2} - \tau\Lambda_1(w,s)\Big\}\ , \tag{4.1.29}$$

Using Eq. (D.14) from Appendix D we differentiate (4.1.29) with respect to $|z|$ which results in

$$\frac{dI_2}{d|z|} = -\frac{1}{2}\tau e^{-|z|} + \frac{\theta(\tau-|z|)}{2}\Big\{-(\tau+|z|)J_0(s) + 2\tau\Lambda_0(w,s)\Big\}\ . \tag{4.1.30}$$

In Eqs. (4.1.29) and (4.1.30), I_2 and its derivative appear to have a term which grows linearly in τ, but this is impossible in view of the integral representations of Eqs. (4.1.8). Using asymptotic expressions for the modified Lommel functions, we shall show in section 4.2 that the large τ dependence is actually an inverse square root.

Writing the phonon contribution as

$$G_p^{SG}(x,x',\tau) = \theta(\tau)\Big\{I_1 + \beta_2 I_2 - \beta_3 sgn(z)\frac{dI_2}{d|z|}\Big\}\ , \tag{4.1.31}$$

we notice that with I_1, I_2 and $\frac{dI_2}{d|z|}$ given by Eqs. (4.1.12), (4.1.29) and (4.1.30), there is a term which does not vanish outside of the "light-cone" (i.e. a term which does not have $\theta(\tau-|z|)$ as a prefactor), namely

$$\theta(\tau)\frac{\tau e^{-|z|}}{2}\Big\{\beta_2 + sgn(z)\beta_3\Big\}\ . \tag{4.1.32}$$

One can show that this term may be rewritten as

$$-\theta(\tau)\tau f^*_{b,1}(x) f_{b,1}(x'). \tag{4.1.33}$$

Hence, when the bound state contribution is added to Eq. (4.1.31) to obtain the full Green function, we are left with an expression which vanishes identically outside of the light-cone:

$$\begin{aligned} G^{SG}(x,x',\tau) &= \frac{\theta(\tau-|z|)}{2}\Big\{J_0(s) + \beta_2[sJ_1(s) - 2\tau\Lambda_1(w,s)] \\ &- \beta_3 sgn(z)[-(\tau+|z|)J_0(s) + 2\tau\Lambda_0(w,s)]\Big\} , \end{aligned} \tag{4.1.34}$$

explicitly demonstrating the retarded boundary conditions applied.

4.1.2 The ϕ^4 Potential

With a slight generalization, the techniques used to evaluate the SG Green function may be applied to the ϕ^4 potential. Proceeding along the same lines, we write the phonon contribution as:

$$G_p^{\phi^4}(x,x',\tau) = \frac{\theta(\tau)}{4}\Big\{\gamma_0 I_0 - \gamma_1 sgn(z)\frac{dI_0}{d|z|} + \gamma_2 I_2 + \gamma_3 sgn(z)\frac{dI_2}{d|z|} + I_4\Big\} , \tag{4.1.35}$$

where I_2 and $\frac{dI_2}{d|z|}$ are given in Eqs. (4.1.29-30) and

$$I_0 = \frac{1}{\pi}\int_0^\infty dk \frac{\cos(|z|k)\sin(\tau\sqrt{1+k^2})}{(1+k^2)^{\frac{3}{2}}(1+4k^2)} , \tag{4.1.36}$$

$$\begin{aligned} I_4 &= \frac{1}{\pi}\int_0^\infty dk \frac{(1+4k^2)\cos(|z|k)\sin(\tau\sqrt{1+k^2})}{(1+k^2)^{\frac{3}{2}}} , &(4.1.37)\\ &= 2\theta(\tau-|z|)J_0(s) - 3I_2 , &(4.1.38) \end{aligned}$$

$$\begin{aligned} \gamma_0 &\equiv 9\{\tanh^2(y)\tanh^2(y') - \tanh(y)\tanh(y')\} , \\ \gamma_1 &\equiv 18\{\tanh(y)\tanh^2(y') - \tanh^2(y)\tanh(y')\} , \\ \gamma_2 &\equiv 9\tanh(y)\tanh(y') - 3\tanh^2(y) - 3\tanh^2(y') , \\ \gamma_3 &\equiv 6\tanh(y) - 6\tanh(y') , &(4.1.39)\\ y &\equiv \frac{x}{2} \quad , \quad y' \equiv \frac{x'}{2} , &(4.1.40) \end{aligned}$$

where Eq. (4.1.12) has been used to simplify Eq. (4.1.37). The remaining integral, I_0, may be reduced by partial fractions to

$$I_0 = \frac{4}{3\pi}\int_0^\infty dk \frac{\cos(|z|k)\sin(\tau\sqrt{1+k^2})}{\sqrt{1+k^2}\,(1+4k^2)} - \frac{I_2}{3} , \tag{4.1.41}$$

$$= \frac{4}{3} I_{01} - \frac{1}{3} I_2 \ , \tag{4.1.42}$$

with I_{01} defined by

$$I_{01} = \frac{1}{\pi} \int_0^\infty dk \frac{\cos(|z|k)\sin(\tau\sqrt{1+k^2})}{\sqrt{1+k^2}\ (1+4k^2)} \ . \tag{4.1.43}$$

To evaluate I_{01} we again substitute $k = \sinh(u)$ which gives us

$$I_{01} = \frac{1}{\pi} \int_0^\infty du \frac{\cos[|z|\sinh(u)]\sin[\tau\cosh(u)]}{1+4\sinh^2(u)} \ , \tag{4.1.44}$$

$$= \frac{1}{2\pi} \int_0^\infty \frac{tdt}{t^4 - t^2 + 1} \sin\Big[at + \frac{b}{t}\Big] \ , \tag{4.1.45}$$

where in going from Eq. (4.1.43) to (4.1.44) substitutions similar to those made in Eqs. (4.1.15-20) have been made. Factoring the denominator of Eq. (4.1.45), we define

$$\beta_\pm^2 = -\, t_\pm^2 = -\beta_\mp = \frac{-1 \mp i\sqrt{3}}{2} \ , \tag{4.1.46}$$

where $t_\pm^2$ are the roots of $t^4 - t^2 + 1$. Using partial fractions, we may write Eq. (4.1.44) as

$$I_{01} = \frac{1}{2\pi i\sqrt{3}} \Big\{ \int_0^\infty \frac{tdt}{t^2+\beta_+^2} \sin\Big[at + \frac{b}{t}\Big] - \int_0^\infty \frac{tdt}{t^2+\beta_-^2} \sin\Big[at + \frac{b}{t}\Big] \Big\} \ , \tag{4.1.47}$$

$$= \frac{-1}{2i\sqrt{3}} [J(\beta_-^2) - J^*(\beta_-^2)] \ , \tag{4.1.48}$$

$$= \frac{-1}{\sqrt{3}} \Im[J(\beta_-^2)] \ , \tag{4.1.49}$$

where $\Im$ denotes the imaginary part and

$$J(\beta^2) = -\ \frac{1}{\pi} \int_0^\infty \frac{tdt}{t^2+\beta^2} \sin\Big[at + \frac{b}{t}\Big] \ . \tag{4.1.50}$$

The integral defined in Eq. (4.1.49) is a slight generalization of Hardy's integrals for Lommel functions [91, 92]. The evaluation of $J(\beta^2)$ follows Hardy's with a few modifications and is presented in Appendix C for completeness. From Eq. (C.21) in Appendix C we have

$$J(\beta_-^2) = \frac{1}{2} e^{-(a\beta_- - \frac{b}{\beta_-})} - \theta(b)\Lambda_2\Big[\frac{2b}{\beta_-}, 2\sqrt{ab}\Big] \ , \tag{4.1.51}$$

$$= \frac{1}{2} e^{-\frac{1}{2}(|z|+i\sqrt{3}\tau)} - \theta(\tau - |z|)\Lambda_2(\beta_+ w, s) \ . \tag{4.1.52}$$

Therefore we have for I_{01} :

$$I_{01} = \frac{1}{2\sqrt{3}} e^{-\frac{|z|}{2}} \sin(\omega_2 \tau) + \frac{\theta(\tau - |z|)}{\sqrt{3}} \Im[\Lambda_0(\beta_+ w, s)] \ , \tag{4.1.53}$$

where

$$\omega_2 \equiv \frac{\sqrt{3}}{2} \ , \tag{4.1.54}$$

and we have used

$$\Im[\Lambda_2(\beta_+ w, s)] = \Im[\Lambda_0(\beta_+ w, s) + J_0(s)] = \Im[\Lambda_0(\beta_+ w, s)] \ . \tag{4.1.55}$$

From Eq. (4.1.35) we see that we need a derivative of I_0, and hence I_{01}, with respect to $|z|$. Using Eq. (D.14) and Eqs. (D.26) from Appendix D we have

$$\frac{dI_{01}}{d|z|} = \frac{-1}{4\sqrt{3}} e^{-\frac{|z|}{2}} \sin(\omega_2 \tau) - \frac{\theta(\tau - |z|)}{2\sqrt{3}} \Im[\Lambda_1(\beta_+ w, s)] \ , \tag{4.1.56}$$

where

$$\frac{\beta_+^2 + 1}{\beta_+} = 1 \ , \tag{4.1.57}$$

has also been used. Collecting all of the pieces, we write for the phonon contribution:

$$\begin{aligned} G_p^{\phi^4}(x, x', \tau) &= \frac{\theta(\tau)}{4} \Big\{ \frac{4}{3}\gamma_0 I_{01} - \frac{4}{3}\gamma_1 sgn(z) \frac{dI_{01}}{d|z|} + [\gamma_2 - \frac{\gamma_0}{3} - 3] I_2 \\ &+ sgn(z)[\frac{\gamma_1}{3} + \gamma_3] \frac{dI_2}{d|z|} + 2\theta(\tau - |z|) J_0(s) \Big\}. \end{aligned} \tag{4.1.58}$$

As in the sine-Gordon case one may show that when we combine the "non-retarded" pieces of the phonon contribution, we get exactly the negative of the bound state contribution; specifically we have

$$\frac{1}{8}[\gamma_2 - \frac{\gamma_0}{3} - 3]\tau e^{-\frac{|z|}{2}} - \frac{sgn(z)}{8}[\frac{\gamma_1}{3} + \gamma_3]\tau e^{-\frac{|z|}{2}} = -\tau f_{b,1}^*(x) f_{b,1}(x') \ , \tag{4.1.59}$$

$$\frac{1}{6\sqrt{3}} e^{-\frac{|z|}{2}} \sin(\omega_2 \tau)\gamma_0 + \frac{1}{12\sqrt{3}} e^{-\frac{|z|}{2}} \sin(\omega_2 \tau) sgn(z)\gamma_1 = -\frac{\sin(\omega_2 \tau)}{\omega_2} f_{b,2}^*(x) f_{b,2}(x'). \tag{4.1.60}$$

With the "non-retarded" portion cancelled by the bound state contribution, we have for the full Green function

$$G^{\phi^4}(x,x',\tau) = \theta(\tau-|z|)\Bigg\{\frac{1}{3\sqrt{3}}\Im[\gamma_0\Lambda_0(\beta_+ w,s)+\frac{1}{2}\gamma_1 sgn(z)\Lambda_1(\beta_+ w,s)]$$
$$+\ \frac{1}{8}[\gamma_2-\frac{\gamma_0}{3}-3][sJ_1(s)-2\tau\Lambda_1(w,s)]$$
$$+\ \frac{sgn(z)}{8}[\frac{\gamma_1}{3}+\gamma_3][-(\tau+|z|)J_0(s)+2\tau\Lambda_0(w,s)]+\frac{1}{2}J_0(s)\Bigg\}\ . \quad (4.1.61)$$

4.1.3 The Double Quadratic Potential

As a final example, we evaluate the DQ Green function. The phonon contribution in this case is

$$G_p^{DQ}(x,x',\tau) = \frac{\theta(\tau-|z|)}{2}\Big\{I_1-\Big[I_2(z_+)-\frac{dI_2(z_+)}{dz_+}\Big]\Big\}\ , \quad (4.1.62)$$

where I_1 is given in Eq. (4.1.12) [with $\mu = 1$] and $I_2(z_+)$ is given in Eq. (4.1.29) with $|z|$ replaced by $z_+ \equiv |x|+|x'|$. Factoring out the non-retarded piece we have

$$\begin{aligned} G^{DQ}(x,x',\tau) &= \frac{\theta(\tau-|z|)}{2}\Big\{J_0(s)-s_+J_1(s_+)+2\tau\Lambda_1(w_+,s_+) \\ &+ (\tau+z_+)J_0(s_+)+2\tau\Lambda_0(w_+,s_+)\Big\}\ , \end{aligned} \quad (4.1.63)$$

with

$$z_+ \equiv |x|+|x'|\ , \quad (4.1.64)$$
$$w_+ \equiv \tau - z_+\ , \quad (4.1.65)$$
$$s_+ \equiv \sqrt{\tau^2-z_+^2}\ . \quad (4.1.66)$$

All three of the Green functions derived above have been checked against numerical integration. Over a large range of values for x, x' and τ, we find agreement to 8 significant digits, which is presently the accuracy of our routines which compute the modified Lommel functions. In addition we have applied the small oscillation operator [see Eq. (4.1.2)] on each of the analytic expressions which, after some tedious algebra, yield the appropriate delta functions. To obtain a final check, we note that by using the orthogonality relation in Eq. (3.1.13) and linear superposition, we see that phonon contribution to the Green functions must be orthogonal to the bound state(s). Numerical integrations confirm this property for all three Green functions.

4.2 Asymptotic Behavior

To obtain asymptotic expressions ($\tau \to \infty$) for the Green functions, we must first find the appropriate limits of the modified Lommel functions. In Appendix E we examine $\Lambda_0(w,s)$ and $\Lambda_1(w,s)$ in the limit as $s \to \infty$ while $w/s \to 1$, which, when w and s are related to τ and z by Eqs. (4.1.27) and (4.1.28), corresponds to $\tau \gg |z|$. This limit is interesting because the expressions for the phonon contributions to the Green functions have a term linear in τ which, in view of the integral expressions, must be cancelled by the other terms.

Since all of the Green functions are expressible in terms of the integrals I_{01}, I_2 and their derivatives with respect to $|z|$ we consider the asymptotic expressions for these quantities first and then combine them to obtain the limits for the Green functions.

To apply the results of Appendix E we must first recast these results in terms of the variables τ and z which are related to w and s by

$$\begin{aligned} w &= \beta(\tau - |z|) , \\ s &= \sqrt{\tau^2 - z^2} , \end{aligned} \tag{4.2.1}$$

where β is either unity or β_+. From Eqs. (E.31) and (E.32) of Appendix E, we have for $\beta = 1$,

$$\begin{aligned} \Lambda_0(w,s) \approx\ & \frac{J_0(s)}{2} + \frac{e^{-|z|}}{2} + \frac{|z|}{2\tau}\sqrt{\frac{2}{\pi s}}\Big\{\cos(s-\frac{\pi}{4})\Big[1+\frac{2R_4(1,\kappa)}{(8s)^2}\Big] \\ & \sin(s-\frac{\pi}{4})\frac{2R_2(1,\kappa)}{8s}\Big\} + O(\tau^{-\frac{9}{2}}) , \end{aligned} \tag{4.2.2}$$

$$\begin{aligned} \Lambda_1(w,s) \approx\ & \frac{e^{-|z|}}{2} - \frac{s}{2\tau}\sqrt{\frac{2}{\pi s}}\Big\{\cos(s-\frac{\pi}{4})\Big[\frac{2[R_2(1,\kappa)-2]}{8s} - \frac{40R_4(1,\kappa)}{(8s)^3}\Big] \\ & -\sin(s-\frac{\pi}{4})\Big[1+\frac{2[R_4(1,\kappa)+12R_2(1,\kappa)]}{(8s)^2}\Big]\Big\} + O(\tau^{-\frac{9}{2}}) , \end{aligned} \tag{4.2.3}$$

where $\kappa \equiv w/s$, R_2 and R_4 are defined in Eqs. (E.29), (E.30), and we have used

$$\epsilon(1,\kappa) = \frac{|z|}{s} \tag{4.2.4}$$

$$\sigma_1(1,\kappa) = \frac{\tau}{2s} , \tag{4.2.5}$$

$$\sigma_2(1,\kappa) = \frac{\tau}{2|z|} , \tag{4.2.6}$$

$$\frac{\sigma_1(1,\kappa)}{\sqrt{1+\epsilon^2(1,\kappa)}} = \frac{1}{2s} \ , \tag{4.2.7}$$

$$\frac{\epsilon(1,\kappa)\sigma_2(1,\kappa)}{1+\epsilon^2(1,\kappa)} = \frac{s}{2\tau} \ . \tag{4.2.8}$$

Inserting the expression for $\Lambda_1(w,s)$ in Eq. (4.2.3) into Eq. (4.1.29), we see that the linear τ dependence exactly cancels (for large τ and $\tau \gg |z|$, both $\theta(\tau-|z|)$ and $\theta(\tau)$ are unity), leaving us with:

$$\begin{aligned} I_2 \approx \ & \frac{sJ_1(s)}{2} + \frac{s}{2}\sqrt{\frac{2}{\pi s}}\Big\{\cos(s-\frac{\pi}{4})\Big[\frac{2[R_2(1,\kappa)-2]}{8s} - \frac{40R_4(1,\kappa)}{(8s)^3}\Big] \\ & -\sin(s-\frac{\pi}{4})\Big[1+\frac{2[R_4(1,\kappa)+12R_2(1,\kappa)]}{(8s)^2}\Big]\Big\} + O(\tau^{-\frac{7}{2}}) \ . \end{aligned} \tag{4.2.9}$$

In Eq. (4.2.9), I_2 now seems to have a $\sqrt{s}$ and therefore $\sqrt{\tau}$ dependence, however this again exactly cancels when $J_1(s)$ is expanded in its asymptotic series resulting in:

$$\begin{aligned} I_2 \approx \ & \frac{1}{2}\sqrt{\frac{2}{\pi s}}\Big\{\sin(s-\frac{\pi}{4})\Big[\frac{15-4[R_4(1,\kappa)+12R_2(1,\kappa)]}{16(8s)}\Big] \\ + \ & \cos(s-\frac{\pi}{4})\Big[\frac{2R_2(1,\kappa)-1}{8} + \frac{5[21/16-R_4(1,\kappa)]}{(8s)^2}\Big]\Big\} + O(\tau^{-\frac{7}{2}}) \end{aligned} \tag{4.2.10}$$

Similarly we have

$$\begin{aligned} \frac{dI_2}{d|z|} \approx \ & \frac{|z|}{2}\sqrt{\frac{2}{\pi s}}\Big\{\cos(s-\frac{\pi}{4})\Big[\frac{9+4R_2(1,\kappa)}{2(8s)^2}\Big] \\ + \ & \sin(s-\frac{\pi}{4})\Big[\frac{2R_2(1,\kappa)-1}{(8s)}\Big]\Big\} + O(\tau^{-\frac{7}{2}}) \ . \end{aligned} \tag{4.2.11}$$

Next we turn to the I_{01} expression which involves modified Lommel functions evaluated at $\beta_+ w$ and s. With $\beta = \beta_+$, $\epsilon(\beta,\kappa)$, $\sigma_1(\beta,\kappa)$ and $\sigma_2(\beta,\kappa)$ become

$$\epsilon(\beta_+,\kappa) = \frac{|z|+i\sqrt{3}\tau}{2s} \ , \tag{4.2.12}$$

$$\sigma_1(\beta_+,\kappa) = \frac{\tau+i\sqrt{3}\ |z|}{4\pi s} \ , \tag{4.2.13}$$

$$\sigma_2(\beta_+,\kappa) = \frac{\kappa}{2s}\frac{(\tau+i\sqrt{3}|z|)(\tau+|z|)}{|z|+i\sqrt{3}\tau} \ . \tag{4.2.14}$$

Inserting Eqs. (4.2.12-14) into Eqs. (E.31) and (E.32), we have

$$\begin{aligned}\Lambda_0(\beta_+ w,s) &\approx \frac{1}{2}e^{-\frac{|z|}{2}}e^{i\omega_2 t}+\frac{1}{2}\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}}\sqrt{\frac{2}{\pi s}}\Big\{\cos(s-\frac{\pi}{4})\big[1+\frac{2R_4(\beta_+,\kappa)}{(8s)^2}\big]\\ &+ \sin(s-\frac{\pi}{4})\frac{2R_2(\beta_+,\kappa)}{(8s)}\Big\}+O(\tau^{-\frac{7}{2}})\ , \qquad (4.2.15)\end{aligned}$$

$$\begin{aligned}\Lambda_1(\beta_+ w,s) &\approx \frac{1}{2}e^{-\frac{|z|}{2}}e^{-i\omega_2 t}-\frac{1}{2}\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}}\sqrt{\frac{2}{\pi s}}\times\\ &\times \Big\{\cos(s-\frac{\pi}{4})\big[\frac{2[R_2(\beta_+,\kappa)-2]}{(8s)}-40\frac{R_4(\beta_+,\kappa)}{(8s)^3}\big]\\ &- \sin(s-\frac{\pi}{4})\big[1+\frac{2[R_4(\beta_+,\kappa)+12R_2(\beta_+,\kappa)]}{(8s)^2}\big]\Big\}\\ &+O(\tau^{-\frac{9}{2}})\ , \qquad (4.2.16)\end{aligned}$$

where we have used

$$\frac{\sigma_1(\beta_+,\kappa)}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}} = \frac{1}{2}\ , \qquad (4.2.17)$$

$$\frac{\epsilon(\beta_+,\kappa)\sigma_2(\beta_+,\kappa)}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}} = \frac{1}{2}\ , \qquad (4.2.18)$$

When Eq. (4.2.15) is inserted into the expression for I_{01}, the oscillatory term in τ cancels leaving us with

$$\begin{aligned}I_{01} &\approx \frac{1}{2\sqrt{3}}\Im\Big\{\frac{1}{\sqrt{1+\epsilon^2(1,\kappa)}}\sqrt{\frac{2}{\pi s}}\big[\cos(s-\frac{\pi}{4})(1+\frac{2R_4(\beta_+,\kappa))}{(8s)^2})\\ &+ \sin(s-\frac{\pi}{4})\frac{2R_2(\beta_+,\kappa)}{(8s)}\big]\Big\}+O(\tau^{-\frac{7}{2}})\ , \qquad (4.2.19)\end{aligned}$$

and

$$\begin{aligned}\frac{dI_2}{d|z|} &\approx \frac{1}{2\sqrt{3}}\Im\Big\{\frac{1}{\sqrt{1+\epsilon^2(1,\kappa)}}\sqrt{\frac{2}{\pi s}}\big[\cos(s-\frac{\pi}{4})\big(\frac{2[R_2(\beta_+,\kappa)-2]}{8s}\big)\\ &- \sin(s-\frac{\pi}{4})\big(1+\frac{2[R_4(\beta_+,\kappa)+12R_2(\beta_+,\kappa)]}{(8s)^2}\big)\big]\Big\}+O(\tau^{-\frac{7}{2}})\ , \qquad (4.2.20)\end{aligned}$$

Now all of the contributions are at hand to obtain, through $O(\tau^{-\frac{7}{2}})$, the asymptotic forms for the Green functions. However, since the expressions are lengthy and not particularly illuminating, we list only the leading terms. Due to the simple analytic form of the bound state contribution, we list only the phonon portions:

$$G_p^{SG}(x,x',\tau) \approx \sqrt{\frac{2}{\pi s}}\Big\{\cos(s-\frac{\pi}{4}) + \frac{1}{8s}\sin(s-\frac{\pi}{4})\Big\} + O(\tau^{-\frac{5}{2}}) \ , \tag{4.2.21}$$

$$\begin{aligned} G_p^{\phi^4}(x,x',\tau) \approx \sqrt{\frac{2}{\pi s}}\Big\{\cos(s-\frac{\pi}{4})\Big[\frac{\gamma_0}{6\sqrt{3}}\Im\Big(\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}}\Big) \\ + \ \frac{1}{8}\Big(\gamma_2 - \frac{\gamma_0}{3} - 3\Big)\Big(\frac{2R_2(1,\kappa)-1}{8}\Big) + 2\Big] \\ - \ \sin(s-\frac{\pi}{4})\Big[\frac{\gamma_1 sgn(z)}{12\sqrt{3}}\Im\Big(\frac{1}{\sqrt{1+\epsilon^2(\beta_+,\kappa)}}\Big)\Big]\Big\} + O(\tau^{-\frac{3}{2}}) \ , \end{aligned} \tag{4.2.22}$$

$$G_p^{DQ}(x,x',\tau) \approx \sqrt{\frac{2}{\pi s}}\cos(s-\frac{\pi}{4}) - \frac{1}{2}\sqrt{\frac{2}{\pi s_+}}\cos(s_+-\frac{\pi}{4})\Big[\frac{2R_2(1,\kappa_+)-1}{8}\Big] + O(\tau^{-\frac{3}{2}}) \ , \tag{4.2.23}$$

where in Eq. (4.2.23), $\kappa_+ \equiv w_+/s_+$.

One may notice that although we have shown that there is no linear τ term in the phonon contributions to the Green functions, the full Green functions have a linear τ term due to the first bound state, namely,

$$\theta(\tau)\tau f_{b,i}^*(x) f_{b,i}(x') \ . \tag{4.2.24}$$

This term may be understood by realizing that when computing the response of a soliton to a perturbation, the effect of this term is to produce a coefficient of the translation mode $f_{b,1}(x)$ which increases with time. Therefore, the soliton will move from its initial position as time progresses. Hence in this case, the linear term is required to describe the translation of the soliton.

The secularity referred to in the introduction is made evident by the linear τ behavior in the coefficient of the translation-mode contribution to the full Green function. Indeed, the use of the full Green function in a perturbation theory of kink dynamics in the presence of external influences is equivalent to the procedure introduced by Fogel et al.[37]. The use of the collective-coordinate method avoids the secularity associated with the translation mode since only the "phonon" part of the Green function is employed (together with the contribution from other bound states, if any ($N \geq 2$)).

4.3 Laplace Transform of the SG Green function

As mentioned in the beginning of this chapter, we can obtain analytic forms for the Laplace transform of the Green functions. In the interest of brevity we present only the transformation for the SG Green functions although the methods below also apply to the other models (ϕ^4 and DQ). From Eq. (4.1.34) we see that Laplace transform of the SG Green function is made up of a sum of the Laplace transform of several Bessel functions plus the Laplace transform of the modified Lommel functions. The Bessel function transforms are easily found in the tables [93] or may be written as derivatives of known transforms and therefore we merely present these results. Defining the Laplace transform of a function $F(\tau)$ to be

$$\bar{F}(\bar{s}) = \mathcal{L}[F(\tau)] \equiv \int_0^\infty d\tau e^{-\bar{s}\tau} F(\tau) \ , \tag{4.3.1}$$

we easily obtain the following:

$$\mathcal{L}\Big[\theta(\tau - |z|) J_0(\sqrt{\tau^2 - |z|^2})\Big] = \frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{\sqrt{\bar{s}^2+1}} \ , \tag{4.3.2}$$

$$\mathcal{L}\Big[\theta(\tau - |z|)\sqrt{\tau^2 - |z|^2} J_1(\sqrt{\tau^2 - |z|^2})\Big] = \left[\frac{1}{\sqrt{\bar{s}^2+1}} + |z|\right]\frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{\bar{s}^2+1} \tag{4.3.3}$$

$$\mathcal{L}\Big[\tau\theta(\tau - |z|) J_0(\sqrt{\tau^2 - |z|^2})\Big] = \left[\frac{1}{\sqrt{\bar{s}^2+1}} + |z|\right]\frac{\bar{s}}{\sqrt{\bar{s}^2+1}}\frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{\sqrt{\bar{s}^2+1}} \tag{4.3.4}$$

With these expressions in hand it remains to compute the Laplace transform of the modified Lommel functions.

4.3.1 Laplace Transform of $\theta(\tau - |z|)\Lambda_n(w, s)$

Recalling the definition for the modified Lommel functions of two variables, we write for $\Lambda_n(w, s)$

$$\Lambda_n(w, s) = \sum_{m=0}^{\infty}\left(\sqrt{\frac{\tau - |z|}{\tau + |z|}}\right)^{n+2m} J_{n+2m}(\sqrt{\tau^2 - |z|^2}) \ . \tag{4.3.5}$$

Since we always have $\tau > |z|$, this sum converges uniformly and therefore in taking the Laplace transform of the sum we can interchange the order of integration and summation. Therefore we are led to consider the Laplace transform of the summand in Eq. (4.3.5) which is found in the tables [94] to be

$$\mathcal{L}\left[\left(\sqrt{\frac{\tau - |z|}{\tau + |z|}}\right)^{n+2m} J_{n+2m}(\sqrt{\tau^2 - |z|^2})\right] = \frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{\sqrt{\bar{s}^2+1}(\sqrt{\bar{s}^2+1} + \bar{s})^{n+2m}} \ . \tag{4.3.6}$$

Therefore

$$\begin{aligned}
\mathcal{L}[\theta(\tau - |z|)\Lambda_n(w,s)] &= \sum_{m=0}^{\infty} \frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{\sqrt{\bar{s}^2+1}} \left[\frac{1}{\sqrt{\bar{s}^2+1}+\bar{s}}\right]^{n+2m} \\
&= \frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{\sqrt{\bar{s}^2+1}} \left[\frac{1}{\sqrt{\bar{s}^2+1}+\bar{s}}\right]^{n-2} \frac{1}{2\bar{s}} \frac{1}{\sqrt{\bar{s}^2+1}+\bar{s}} \qquad (4.3.7)
\end{aligned}$$

where in evaluating the sum I have used the fact that when doing an inverse Laplace transform, $\Re(\bar{s}) > 0$ and therefore the sum converges uniformly. Now all of the components are at hand to obtain the expression for the Laplace transform of the SG Green function. In doing the algebra, quite a bit of cancellation occurs leaving us with a remarkably simple expression for the Laplace transform

$$\begin{aligned}
\bar{G}^{SG}(x,x';\bar{s}) &\equiv \mathcal{L}[G(x,x',\tau)] \\
&= \frac{e^{-|z|\sqrt{\bar{s}^2+1}}}{2}\left\{\frac{1}{\sqrt{\bar{s}^2+1}} - \frac{\beta_2}{\bar{s}^2\sqrt{\bar{s}^2+1}} - \frac{\beta_3 sgn(z)}{\bar{s}^2}\right\}. \qquad (4.3.8)
\end{aligned}$$

4.3.2 Bromwich Representation for $\psi(x,t)$

The derivation of the Laplace transform is not merely an academic execise as it may prove useful for the numerical evaluation of the phonon field. To see that this is the case we now substitute the inverse Laplace representation of the SG Green function into the integral expression for ψ given in Eq. (3.4.12)

$$\begin{aligned}
\psi(x,t) &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\bar{s}\ e^{\bar{s}\tau} \bar{G}^{SG}(x,x';\bar{s}) I(x',t') \qquad (4.3.9) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dt'\ \bar{G}^{SG}(x,x';c+iy) e^{-(c+iy)(t'-t)} I(x',t') \qquad (4.3.10)
\end{aligned}$$

where c is a positive real constant which is greater than 0 (i.e. this is the real part of the "right-most" pole of $\bar{G}^{SG}(x,x';\bar{s})$). To make further analytic progress, we consider a specific perturbation, namely we choose a linear coupling function $F = \Phi$ and a time-independent perturbation which is well localized in space

$$v(x) = \lambda\left\{e^{-(x-x_0)^2} - e^{-(x+x_0)^2}\right\}, \qquad (4.3.11)$$

(this is one of the perturbations examined in Chapter 5). In this case the inhomogeneous term $I(x',t')$ may be written as

$$I(x',t') = \psi_0(x'+X(t'))\mathrm{sech}^2(x') - \frac{\phi_c'(x')}{M_0} \int_{-\infty}^{\infty} dz\ \phi_c'(z)\psi_0(z+X(t'))sech^2(z), \qquad (4.3.12)$$

where the "background response" field ψ_0 satisifies

$$-\psi_0'' + \psi_0 = v(x) \ . \tag{4.3.13}$$

We can solve for ψ_0 by Fourier transforming and to this end we introduce the following inverse transforms

$$\begin{aligned} \bar{\psi}_0(k) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dx\ e^{ikx} \psi_0(x) \ , & (4.3.14) \\ \bar{v}(k) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dx\ e^{ikx} v(x) \ . & (4.3.15) \end{aligned}$$

By substituting these transforms into Eq. (4.3.13) we see that they are related by

$$\bar{\psi}_0(k) = \frac{\bar{v}(k)}{1+k^2} \ . \tag{4.3.16}$$

Next we consider the t' integral in Eq. (4.3.10)

$$\int\limits_{-\infty}^{\infty} dt' e^{-(c+iy)t'} I(x', t') \ . \tag{4.3.17}$$

From Eq. (4.3.12) we see that the only t' dependence occurs through $X(t')$. To lowest order we approximate this by

$$X(t') \approx X_0 + V_0 t' \ , \tag{4.3.18}$$

where $X_0 \equiv X(0)$ and $V_0 \equiv \dot{X}(0)$. By assuming Eq. (4.3.18) to be valid, we restrict ourselves to the study of the case in which the kink scatters off the perturbation to ∞. We are therefore led to consider the integral

$$\begin{aligned} J(\xi, y; c) &\equiv \int\limits_{-\infty}^{\infty} dt' e^{-(c+iy)t'} \psi_0(\xi + X_0 + V_0 t') & (4.3.19) \\ &= \frac{e^{i(\xi+X_0)(y-ic)/V_0}}{V_0} \int\limits_{-\infty}^{\infty} d\zeta\ e^{-i\zeta(y-ic)/V_0} \psi_0(\zeta) \ , & (4.3.20) \end{aligned}$$

where ξ is either x' or z as required by Eq. (4.3.12). Although the integrand seems to diverge as $\zeta \to -\infty$ it does not since one can show that for the perturbation chosen we have

$$\bar{v}(k) \approx e^{-k^2} \tag{4.3.21}$$

and therefore

$$\psi_0(x) \approx \int_{-\infty}^{\infty} dk \frac{e^{-k^2} e^{-ikx}}{1+k^2} \ , \tag{4.3.22}$$

and hence $\psi_0(x)$ will decay faster than e^{-x^2}. Since the integral converges we may analytically continue it and obtain the result

$$J(\xi, y; c) = \sqrt{2\pi} \frac{e^{i(\xi+X_0)(y-ic)/V_0}}{V_0} \bar{\psi}_0\Big(\frac{y-ic}{V_0}\Big) \ . \tag{4.3.23}$$

Having carried out the above integration we return to the expression (4.3.12) for $I(x', t')$ and find that this integral occurs inside the spatial integral over z and therefore we consider the integral

$$\frac{\sqrt{2\pi}}{V_0} e^{iX_0(y-ic)/V_0} \bar{\psi}_0\Big(\frac{y-ic}{V_0}\Big) \int_{-\infty}^{\infty} dz \ \phi_c'(z) \mathrm{sech}^2(z) e^{iz(y-ic)/V_0} \ . \tag{4.3.24}$$

Again it seems that this integral

$$\int_{-\infty}^{\infty} dz \ \mathrm{sech}^3(z) e^{iz(y-ic)/V_0} \tag{4.3.25}$$

may not converge due to the factor of e^{zc/v_0} (I have used the fact that $\phi_c'(z) = 2\mathrm{sech}(z)$ for SG). Since c needs only to be > 0, we can choose it such that

$$\frac{c}{V_0} < 3 \tag{4.3.26}$$

so that the $\mathrm{sech}^3(z)$ factor will dominate. Using the fact that the integral does indeed exist, we analytically continue a standard result from the Tables [95] to obtain

$$\int_{-\infty}^{\infty} dz \ \mathrm{sech}^3(z) e^{iz(y-ic)/V_0} = \pi \left[\left(\frac{4-ic}{V_0} \right)^2 + 1 \right] \mathrm{sech}\Big[\frac{\pi(y-ic)}{2V_0} \Big] \ . \tag{4.3.27}$$

Using all of these pieces we can write another integral expression for the phonon field

$$\begin{aligned} \psi(x,t) \ &= \ \frac{e^{ct} e^{cX_0/V_0}}{V_0 \sqrt{2\pi}} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy \bar{G}^{SG}(x, x'; c+iy) \frac{e^{iyt}}{\cosh x'} \bar{\psi}_0\Big(\frac{y-ic}{V_0}\Big) \times \\ &\times \ \left\{ \frac{e^{ix'(y-ic)/V_0}}{\cosh(x')} - \frac{2\pi}{M_0} \Big[\Big(\frac{y-ic}{V_0} \Big)^2 + 1 \Big] \mathrm{sech}\Big[\frac{\pi(y-ic)}{2V_0} \Big] \right\} \ . \end{aligned} \tag{4.3.28}$$

Again we must evaluate a two-dimensional integral to obtain values for ψ, however this integrand has a rapidly decaying factor, namely the Laplace transform of the Green function. However, the exponential factor

$$e^{ix'(y-ic)/V_0} \tag{4.3.29}$$

still oscillates rapidly in y since typical values of X_0 and V_0 are -10 and .3 respectively. In addition we have an exponential term in time which also yields rapid oscillations for large t. However this type of oscillating behavior may prove to be the key to a quick and efficient evaluation of Eq. (4.3.28). The key lies in the fact that the evaluation of Eq. (4.3.28) is written in "Fourier transform form". The fact that the integrand in Eq. (4.3.28) already involves Fourier transform of the ψ_0 field and therefore the possibility of using the convolution theorem exists. If nothing else, we have in Eq. (4.3.28) an approximate analytic expression (in terms of an integral) of the time Fourier transform for the ψ field

Assuming that Fourier transform methods are not tractable, the ease with which Eq. (4.3.28) is evaluated depends on whether the oscillations are damped quickly enough by the decaying factors. In addition to the rapid decay caused by the Laplace transform of the Green function, the Fourier transform of the background field is also rapidly decaying. For the perturbation examined in this section, we have the following analytic form for $\bar{\psi}_0$:

$$\bar{\psi}_0 = \frac{2\sqrt{\pi}\cos kx_0 e^{-k^2/4}}{1+k^2}\ . \tag{4.3.30}$$

With this additional decaying factor, it is quite possible that this integral may be done numerically. One of the major problems with the previous two-dimensional integral expression for the ψ field is that one had to evaluate the Green function itself, which involves calculating and summing approximately 200 Bessel functions. Even when these codes are vectorized and run on a Cray-1 computer, these manipulations require quite a bit of time. In view of the problems encountered with the numerical integration of the ψ PDE (see §5.2), numerical evaluation of ψ using Eq. (4.3.28) is a very attractive possibility which is currently under investigation.

4.4 Representative plots

To illustrate the behavior of the Green functions we present several plots of the *phonon* part of the SG Green function [plots for the other Green functions derived look very similar] . The numerical values for these plots are easily obtained from the formulae in Appendix E.

We can get a feel for how the Green functions should behave by recalling that $G(x, x', t-t')$ represents the response of the field at (x, t) due to a delta function source at (x', t'). To make this more concrete we can imagine striking one of the pendula of the sine-Gordon pendulum chain with a sharp blow and watching the response of the other pendula. We expect to see a pulse move out from the "hit pendulum" and propagate toward the ends of the chain. In Figure 4.1 we plot the Green function vs. x and x' for various values of $t = \tau$ (we have chosen $t' = 0$). Fixing $x' = 8$ (i.e. the pendulum at $x = 8$ is struck) in Figure 4.1a, we move in the direction of increasing x, starting at $x = 0$. Until x is on the order of 2, $G(x, x', \tau)$ is zero, meaning that the disturbance has not yet had enough time to propagate from $x = 8$ to $x < 2$ (or $x > 14$). For $\tau = 4$, time has progressed (recall we have fixed $t' = 0$) and the disturbance has propagated further outwards. At $t = 8$ the pulse reaches $x = 8$. In Figures 4.1e to 4.1h the pulse has propagated off the scales, leaving behind "ripples". As τ further increases the amplitude continues to decrease in accord with the asymptotic behavior derived in section 4.2.

If one were to follow the procedure outlined in the preceding paragraph with $x' = 3$, one would note that before the pulse arrives at a particular position, the Green function is not zero. This is because we have plotted the phonon contribution, which has a non-retarded part which exactly cancels the bound state contribution. It is this non-retarded part which gives a non-zero value for the phonon contribution to the Green function "before the pulse arrives". We see this only near $x = x' = 0$ because the bound state contribution is proportional to $e^{-|z|}$ [SG], $\mathrm{sech}(x)\mathrm{sech}(x')$ [ϕ^4] or $e^{-|x|}e^{-|x'|}$ [DQ].

Since the computation of the phonon response ψ involves integrals of the Green function over x' and t', it is interesting to see the behavior of $G(x, x', t - t')$ for fixed x and t. In Figure 4.2 we plot the sine-Gordon Green function for $x = 25$ and $t = 50$. One of the interesting features is the step function which represents the "light cone". In performing the numerical integrals one must be careful not to integrate through this step function since most numerical integrators cannot handle such discontinuities. Another feature which presents some numerical difficulties is the oscillation in time. Of course this oscillation will represent problems only if we must integrate over several of these periods (which is in fact the case for the perturbations examined in Chapter 5).

In Figure 4.3 we present illustrates one of the asymptotic limits of the Green functions. The fact that the Green functions are not functions solely of $x - x'$ is a consequence of the broken translational invariance which results from the introduction of a kink. The only dependence on x and x' which is not through the combination $x - x'$ enters through the functions β_i (SG) and γ_i (ϕ^4). All of these functions depend on x and x' through various combinations of $\tanh(x)$ and $\tanh(x')$. For both x and x' large these β and γ factors are constants so one expects that for both x and x' large the Green functions should depend only on $x - x'$. This fact is illustrated by the plot in Figure 4.3. One can understand this fact analytically by recalling that the functions $f_k(x)$ which are used to define the

Figure 4.1: The time evolution for the phonon contribution to the SG Green function $G(x, x', t - t')$ in the $x - x'$ plane.

Figure 4.2: The phonon contribution to the SG Green function $G(x, x', t - t')$ in the $x' - t'$ plane.

Figure 4.3: The SG Green function $G(x, x', t - t')$ in the $x - x'$ plane. Note the reflection symmetry about the $x = x'$ line.

Green functions are asymptotically plane waves for large x and hence this behavior is to be expected. This behavior may prove useful for certain perturbations if one must perform integrals only over this translationally invariant region.

Figures 4.4 and 4.5 show plots of the real part of the Laplace transform of the sine-Gordon Green function. In Figure 4.4 we plot the real part of the Laplace transform $\bar{G}^{SG}(x, x', \bar{s})$ vs. x and x' for fixed $\bar{s} = 2+2i$. Here we see the dominance of the exponential factor $e^{-|z|\sqrt{\bar{s}^2+1}}$ in Eq. (4.3.8) since the modulus of $\bar{s}$ is large enough so that the factors which do not depend on $x - x'$, that is the factors involving β_2 and β_3, are small compared with the first term in Eq. (4.3.8). The rapid decay in x' shown in Figure 4.4 makes the integral in Eq. (4.3.28) converge rapidly. One might think that the cusp shown in this figure would pose a problem when Eq. (4.3.28) is numerically evaluated. However, one must realize that the integral in Eq. (4.3.28) is not over the $x - x'$ plane but over the $x' - \bar{s}_i$ plane where $\bar{s}_i$ is the imaginary part of the Laplace transform variable. To get a feel for the dependence on the Laplace transform variable $\bar{s}$, we plot in Figure 4.5 the real part of the Laplace transform $\bar{G}^{SG}(x, x'; \bar{s})$ in the complex $\bar{s}$ plane for $x = 2.0$ and $x' = 1.0$. The interesting feature in this plot is the dependence on the imaginary part of $\bar{s}$ which is a rapid decay. Again this is not surprising since the analytic expression given in Eq. (4.3.8) involves an exponential factor of the form

$$e^{-|z|\sqrt{\bar{s}^2+1}} \, .$$

Since the Bromwich integral for the ψ field involves integrating in the complex $\bar{s}$ plane along a line parallel to the imaginary $\bar{s}$ axis, this rapid decay should greatly facilitate the numerical calculations.

Figure 4.4: The real part of the Laplace transform of the sine-Gordon Green function $\bar{G}^{SG}(x, x'; \bar{s})$ plotted vs. x and x' for $\bar{s} = 2 + 2i$.

Figure 4.5: The real part of the Laplace transform of the sine-Gordon Green function $\bar{G}^{SG}(x, x'; \bar{s})$ plotted in the complex $\bar{s}$ plane for $x = 2.0$ and $x' = 1.0$.

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