Chapter 7

Kink-Antikink Collisions in ϕ^4

So far we have dealt with the use of a single collective coordinate which represents the center of mass of a Klein-Gordon kink. Because we have a canonical transformation from the original field variables to the "kink" variables, this method is on very firm ground and therefore is expected to yield reasonable results. However, the rigor lent by this transformation is also a weakness since one cannot expect to find the appropriate canonical transformation for an arbitrary system (assuming one even exists). Since this canonical transformation is based on the physically reasonable decomposition

$$\Phi(x,t) = \phi_c(x - X(t)) + \chi(x - X(t), t) \ ,$$

one would hope that similar physically reasonable ansätze which do not necessarily represent canonical transformations would also prove useful. In order to explore this possibility, we consider in this chapter the use of two collective coordinates to model the kink-antikink collisions in ϕ^4 field theory. We begin by reviewing the behavior observed in the numerical simulation of the PDE. In section 7.2 we outline collective coordinate approaches which have been used and introduce an ansatz based on two collective coordinates. Section 7.3 contains plots and limiting analytic forms for the coordinate-dependent masses and potential which one obtains from the averaged Lagrangian. The equations of motion are presented and their numerical solution discussed in section 7.4. These solutions of the equations indicate that the ansatz breaks down when the two kinks collide. The limiting case in which the kinks are very close is examined in section 7.5 in which we find that one of the coordinates undergoes very rapid changes as the separation between the kinks goes to zero. A new ansatz based on the original one but which includes "relativistic" terms is proposed in section 7.6. Simulations of the equations of motion which result from this ansatz are currently being carried out.

7.1 Observed Phenomena in Numerical Simulations

As mentioned in the introduction, the ϕ^4 system is not integrable but it does possess exact kink(+) and antikink(-) solutions

$$\phi_K(x) = \pm \tanh \frac{(x - x_0)}{m\sqrt{2}} \ . \tag{7.1.1}$$

Since the ϕ^4 system is not integrable but possesses solitary wave solutions (not solitons), it of interest to study the interaction between two such solitary waves. Several investigators [125, 126, 127, 128, 129, 130, 131, 132] have studied the collision of a kink and antikink by the direct numerical integration of the PDE. Initial studies showed that when the kink velocities were above some critical value v_c, the kinks scattered off of one another inelastically transferring energy to other modes of the system such as radiation ("phonons"). For velocities less than v_c the kinks were found to form a bound state, again transferring some energy into the radiation degrees of freedom. Further investigation showed that for certain velocity intervals $v_i < v < v_j < v_c$ (see Figure 7.1) the kinks did not form a bound state but scattered off to $\pm\infty$. Similar phenomena have been observed in other nonintegrable systems such as the parametrically modified sine-Gordon [50] and double sine-Gordon [133] systems. These "resonance windows" have been quantitatively explained by Campbell et al. [15] in terms of an exchange of energy between the kink translational energy and a localized mode known as the "shape mode", which can be thought of as representing a modification of the kink solution. The basic idea is that when the kinks first collide, there is an energy transfer into the shape mode. The kinks then move apart, but not having enough energy to overcome the attractive potential which exists between them (i.e. some energy was given to the shape mode), they fall back toward one another. When they collide again, the energy in the shape mode can be transferred back to the translational motion if the time between the collisions obeys the following resonance condition:

$$\omega_2 T = \delta + 2n\pi \ ,$$

where $\omega_2 = \sqrt{3}/2$ is the frequency of the shape mode. Such a transfer of energy allows the kink and antikink to overcome the attractive potential and escape to infinity. Using these ideas Campbell et al. have been able to predict the bounds v_i of the resonance windows which are in good agreement with the results obtained from the numerical simulations. The analysis, however treats the collision as a "black box" and does not provide any details of the collision as do the PDE simulations.

Figure 7.1: Results of a numerical simulation showing the final kink velocity after a ϕ^4 $K\bar{K}$ collision as a function of the initial kink velocity. A final velocity of zero indicates the formation of a bound state. Taken from Ref. 15 with the permission of the authors.

7.2 Collective Coordinate Approaches

To gain an understanding of the collision process without solving the PDE, several collective coordinate methods have been put forth to study the $K\bar{K}$ collisions. Although Aubry [130] was the first to observe the resonance structure, Kudryavtsev [125] was the first to implement coordinates which showed that the effective potential between the kink and antikink was attractive. In another early study, Sugiyama [131] introduced collective coordinates which represent the center of mass of the kinks and the amplitudes of the shape mode and radiation degrees of freedom. His analysis was purely analytic, producing an attractive potential in which the kinks moved and a solution for the shape mode coordinate which exhibited harmonic oscillations.

A collective coordinate ansatz very similar to that used by Sugiyama was introduced by Jeyadev and Schrieffer [134]. In the notation established in section 3.1 the ansatz has the form

$$\begin{aligned}\Phi_A(x,t) &= 1+\tanh y_- - \tanh y_+ \\ &+ A(t)\left[f_{b,2}(y_-)\cos\frac{\omega_{b,2}(t-\beta x)}{\sqrt{1-\beta^2}} - f_{b,2}(y_+)\cos\frac{\omega_{b,2}(t+\beta x)}{\sqrt{1-\beta^2}}\right] \\ &+ \sum_k B_k(t)\left[f_k(y_-)\cos\frac{\omega_k(t-\beta x)}{\sqrt{1-\beta^2}} - f_k(y_+)\cos\frac{\omega_k(t+\beta x)}{\sqrt{1-\beta^2}}\right] \quad (7.2.1)\end{aligned}$$

with the definitions

$$y_\pm = \frac{x\pm\alpha(t)}{\sqrt{2}\sqrt{1-\beta^2}} \quad , \quad \beta=\dot{\alpha}\ . \quad (7.2.2)$$

Substitution of this ansatz into the Lagrangian density and integration over space yields a Lagrangian which depends on the collective coordinates $\alpha(t), A(t), B_k(t)$ and their time derivatives. The resulting equations of motion are quite complex when all of the relativistic and phonon terms are included and therefore only the lowest order terms in the velocity were included in the simulations, the phonon terms being dropped entirely. The numerical results based on this model showed that the kinks attained relativistic velocities (in fact β became > 1). When this occurred the amplitude of the shape mode also became large, which in turn caused the velocity to diverge.

A slightly different ansatz

$$\Phi_A(x,t) = \frac{m}{\sqrt{\lambda}}\left\{1-\tanh\Big[\frac{my_0(x-x_0)}{\sqrt{2}}\Big]+\tanh\Big[\frac{my_0(x+x_0)}{\sqrt{2}}\Big]\right\}\ . \quad (7.2.3)$$

was put forth by Campbell et al. [15]in their original work describing the PDE simulations . This ansatz represents a kink-antikink pair moving in opposite directions according to the center of mass variable $x_0(t)$ (see Figure 7.2). Like the

Figure 7.2: Schematic representation of the ansatz in Eq. (7.2.3)

previous ansätze, there is a collective coordinate x_0 which describes the center of mass motion of the kinks. The y_0 coordinate takes the place of the shape mode contribution by allowing the width of the kinks $(1/y_0)$ to vary as a function of time. The x_0 collective coordinate is much like the X coordinate used in the previous chapter in that it is a result of the translational invariance of the original equations. The y_0 coordinate, however appears merely as a parameter in the ansatz. Equation (7.1.1) is a solution of the field equations only for $y_0 = 1$, x_0 being able to take on any value. Therefore the y_0 coordinate has been termed a "parametric collective coordinate" [42, 15] while the x_0 coordinate is a "linear collective coordinate". No matter what the values of x_0 and y_0, the ansatz given in Eq. (7.2.3) is not an exact solution of the original field equations and it does not represent a canonical transformation to a new set of variables; however in view of the explanation of the resonance windows given by Campbell et al., it is certainly a reasonable choice.

Proceeding along the same lines as Jeyadev and Schrieffer [134] , we substitute this ansatz into the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\Big(\frac{\partial_t \Phi_A}{\partial t}\Big)^2 - \frac{1}{2}\Big(\frac{\partial_x \Phi_A}{\partial t}\Big)^2 + \frac{\lambda}{4}(\Phi_A^2 - \frac{m^2}{\lambda})^2 \ , \tag{7.2.4}$$

and integrate over x, yielding a Lagrangian

$$\begin{aligned} L(x_0, \dot{x}_0, y_0, \dot{y}_0) &= \frac{1}{2} m_1(x_0, y_0)\dot{x}_0^2 + m_2(x_0, y_0)\dot{x}_0\dot{y}_0 + \frac{1}{2} m_3(x_0, y_0)\dot{y}_0^2 \\ &- V(x_0, y_0) \ , \end{aligned} \tag{7.2.5}$$

where the expressions for the masses and potentials along with some useful limits are given in Appendix I. One of the interesting features of this Lagrangian and the associated Hamiltonian is the appearance of coordinate dependent masses. Since the masses depend on the coordinates, we cannot use the usual potential energy arguments to give us an idea of what the solution is. One might ask whether one can somehow make a transformation to a new set of variables x_0', y_0' in terms of which the masses are coordinate independent. Such a procedure is available for systems with only one coordinate [135] and merely requires finding the transformation to a new variable $q(q')$ such that

$$m(q)\dot{q}^2 = \tilde{m}\dot{\tilde{q}}^2 \ . \tag{7.2.6}$$

This equation is easily integrated to yield

$$\tilde{q}(q) - \tilde{q}(q_0) = \int_{q_0}^{q} dq' \sqrt{\frac{m(q')}{\tilde{m}}} \ . \tag{7.2.7}$$

The analogous procedure for our system involves first diagonalizing the kinetic energy terms in Eq. (7.2.5), followed by the integration of a coupled set of ODEs.

Since we are not assured of finding a solution of these equations (even numerically), this method shows little promise. Since we have to deal with the coordinate dependent masses and potentials, it is helpful to become acquainted with this dependence.

7.3 The Potential $V(x_0, y_0)$ and Masses $m_i(x_0, y_0)$

As mentioned above, the effective mutual potential experienced by the kink and antikink is attractive as is shown in Figure 7.3. One might argue that since the masses depend on the coordinates, interpreting the effective potential as attractive is not valid; however one can always restrict the coordinates to a small enough range so that the masses are essentially constant. The linear behavior in the potential for values of $x_0 < 0$ represent the fact that the kinks cannot pass through one another to infinity. One can understand this by considering the field amplitude for a configuration in which the kinks have passed through one another (see Figure 7.2). In the segment of length L, the field amplitude is a constant ϕ_0. The energy content of this segment is $(\phi_0^2 - 1)^2 L$ which diverges linearly with L as shown in Figure 7.3.

The masses $m_1(x_0, y_0), m_2(x_0, y_0)$ and $m_3(x_0, y_0)$ are plotted in Figures 7.4-7.6. (It should be noted that in all of the plots, the minimum value of y_0 plotted is not 0. The scale begins at $y_0 = 0$ because the graphics package used automatically scales the plots so that the numbers on the scale are "round" numbers.) For large x_0, m_1 approaches a constant value which is easily shown to be (see Appendix I)

$$m_1(x_0, y_0) \longrightarrow \frac{8}{3}\sqrt{2}m^3 y_0/\lambda \ , \tag{7.3.1}$$

which is twice the mass of a single kink. Figure 7.5 shows that for small x_0, m_2 vanishes linearly while m_3 vanishes quadratically. One of the consequences of these vanishing masses is that for $x_0 = 0$ the kinetic energy is entirely carried by the translation of the kinks. More importantly, we see from the Lagrangian that for m_2 and m_3 equal zero, the value of $\dot{y}_0$ is arbitrary, a fact that will give rise to numerical trouble when the equations of motion are integrated. Finally we note that the mass m_3 shows a divergence as the y_0 coordinate approaches zero. All of the limiting properties of the potential and masses are summarized in Table 7.1.

From the analytic expressions for the masses and potentials given in Appendix I one might expect that as either the x_0 or y_0 coordinates tends to zero, these quantities might not be computed accurately since both the numerator and denominator tend to zero. To avoid such problems the limits of the potentials and masses were taken analytically. With the aide of the symbolic manipulation program MACSYMA [136] the Taylor series were taken up to and including terms

Figure 7.3: The effective potential $V(x_0, y_0)$.

Figure 7.4: The mass $m_1(x_0, y_0)$.

Figure 7.5: The mass $m_2(x_0, y_0)$.

Figure 7.6: The mass $m_3(x_0, y_0)$.

	$z_0 \to 0$	$z_0 \to \infty$
$V_1(x_0, y_0)$	$\frac{8\sqrt{2}m^5 x_0^2 y_0^3}{15\lambda}$	$\frac{2\sqrt{2}m^3 y_0}{3\lambda}$
$V_2(x_0, y_0)$	$\frac{8\sqrt{2}m^5 x_0^2 y_0}{3\lambda}$	$\frac{2\sqrt{2}m^3}{3\lambda y_0}$
$m_1(x_0, y_0)$	$\frac{8\sqrt{2}m^3 y_0}{3\lambda}\left[1 - \frac{2m^2 x_0^2 y_0^2}{5}\right]$	$\frac{8\sqrt{2}m^3 y_0}{3\lambda}$
$m_2(x_0, y_0)$	$\frac{8\sqrt{2}m^3 x_0}{3\lambda}\left[\frac{1}{2} - \frac{2m^2 x_0^2 y_0^2}{5}\right]$	0
$m_3(x_0, y_0)$	$\frac{8\pi^2\sqrt{2}m^3 x_0^2}{45\lambda y_0}$	$\frac{8m}{\sqrt{2}\lambda y_0^3}\frac{1}{3}\left(\frac{\pi^2}{3} - 1\right)$

Table 7.1: Limiting values for the potentials and masses for z_0 approaching 0 and ∞. The total potential V is the sum of V_1 and V_2. Analytic expressions for V_1 and V_2 are given in Appendix I.

on the order of z_0^{10} with $z_0 = mx_0y_0/\sqrt{2}$. To assure a smooth transition from the analytic to Taylor series expressions both quantities were computed for a variety of small values of z_0. For z_0 of the order of 0.01, the expressions gave the same values to *at least* 9 significant digits. As a further check on the analytic forms given in Appendix I, the integral expressions were numerically evaluated. Again we found the analytic and numerically integrated values of the masses and potential agree to *at least* 9 significant digits.

7.4 Equations of Motion

Application of the Euler-Lagrange method yields the following equations of motion for x_0 and y_0:

$$\begin{aligned} m_1\ddot{x}_0 + m_2\ddot{y}_0 &+ \frac{1}{2}\frac{\partial m_1}{\partial x_0}\dot{x}_0^2 \\ &+ \frac{\partial m_1}{\partial y_0}\dot{x}_0\dot{y}_0 + \frac{\partial m_2}{\partial y_0}\dot{y}_0^2 - \frac{1}{2}\frac{\partial m_3}{\partial x_0}\dot{y}_0^2 + \frac{\partial V}{\partial x_0} = 0 \ , \end{aligned} \tag{7.4.1}$$

$$m_3\ddot{y}_0 + m_2\ddot{x}_0 + \frac{1}{2}\frac{\partial m_3}{\partial y_0}\dot{y}_0^2$$

Figure 7.7: Position (solid) and velocity (dashed) for a "wobbling kink" solution.

$$+ \quad \frac{\partial m_3}{\partial x_0}\dot{x}_0\dot{y}_0 + \frac{\partial m_2}{\partial x_0}\dot{x}_0^2 - \frac{1}{2}\frac{\partial m_1}{\partial x_0}\dot{x}_0^2 + \frac{\partial V}{\partial y_0} = 0 \ . \qquad (7.4.2)$$

In general this set of coupled equations must be solved numerically; however, an analytic solution can be found in the limiting case in which x_0 approaches ∞. Guided by the numerical integration of the PDE, one is led to search for a solution in which the velocity of the kinks oscillates about a constant value. Such a solution has been obtained by Campbell [137], with the period of oscillation given by

$$T = 2\pi\sqrt{\Big(\frac{\pi^2}{6} - 1\Big)\Big(1 - v_f^2\Big)} \approx 5.04\sqrt{1 - v_f^2} \ , \qquad (7.4.3)$$

where v_f is the mean of the final velocity. This analytic result proves to be a good check of the numerical integrator. In Figure 7.7 we present results of the numerical integration of Eqs. (7.4.1-2) for initial conditions

$$x_0 = 20 \quad , \quad \dot{x}_0 = 0.2 \quad , \quad y_0 = 1 \quad , \quad \dot{y}_0 = 0 \ . \qquad (7.4.4)$$

Making a rough measurement from this graph we find that the oscillation period is 4.94 compared with 4.94 as computed from Eq. (7.4.3) with v_f approximated by 0.196.

In the above example we started the kink and antikink moving away from each other so that a comparison could be made with analytic results (initial conditions in which the kink and antikink collided yield simliar results). Figure 7.8 shows the integrated values of the variables and their time derivatives for initial conditions for which the kink and antikink collide. Initially the kink and antikink travel toward each other with initial velocities of -0.2 and +0.2 respectively. When the kink and antikink are approximately 3 units apart they begin to accelerate towards one another under the influence of their mutually attractive potential. At x_0 the the kinks very abruptly bounce off of one another after which they move apart, their velocities experiencing small oscillations which represent a transfer of energy into the oscillating width (shape mode) of the kinks (see Figure 7.8). From Figure 7.1 we see that for the initial velocity of -0.2, the kinks should indeed eventually scatter to $\pm\infty$; however the numerical simulations of the PDE indicate that the kink and antikink actually pass through one another ($x_0 < 0$) before they turn around and move off to $\pm\infty$. Furthermore, before they separate to $\pm\infty$, they should experience a second collision in which the energy given to the shape mode oscillation is returned to the translational motion allowing them to escape. This asumes that the ansatz will capture all of the details of the collision which one cannot expect since we allow for no radiation degrees of freedom. However, one would hope to be able to capture the resonance windows. Additional simulations with initial velocities which are not in the windows, that is, initial velocities for which we should see a bound state formed, also show this type of hard bounce. Finally in Figure 7.8 we see the y_0 coordinate is well behaved until it increases rapidly when x_0 approaches 0. The oscillations which occur after this sharp spike again reflect the sharing of energy between the translational kinetic energy and the energy associated with the changing kink width.

In Figure 7.9 we examine more closely the region for which the "hard bounce" seen in Figure 7.8 occurs. The initial conditions used for this run correspond to the values of the variables and their derivatives at $t = 42$ in Figure 7.8. Here we see that the hard bounce at $x_0 = 0$ in fact occurs smoothly on a smaller time scale. Examination of the plots of the y_0 coordinate and its derivative on this finer time scale are further causes of concern since a y_0 value of 40 represents extremely sharp kinks, another feature which is not observed in the PDE simulations. Another interesting feature of these plots is that the kink velocities approach -1 and then turn around, echoing the results of Jeyadev and Schrieffer [134]. The fact that the velocity gets so close to its relativistic limit of -1 seems to indicate that a "relativistic" treatment of the problem is in order. This possibility is outlined in section 7.6.

Given these rather unexpected and somewhat unphysical results, one immediately questions the accuracy of the codes used to integrate the equations. We have already mentioned in section 7.3 that extreme care has been taken in the eval-

Figure 7.8: The kink position x_0 (solid) and inverse width y_0 (solid) along with their time derivatives (dashed) as a function of time.

Figure 7.9: Blow up of the “hard bounce” region of Figure 7.8

Figure 7.10: Hamiltonian vs. time during the bounce.

uation of the potential, masses, and their derivatives, and therefore they can be ruled out as a possible problem. Next one questions the accuracy of the numerical integrator used. Since a Hamiltonian exists for this problem, namely

$$\begin{aligned} H(x_0, \dot{x}_0, y_0, \dot{y}_0) &= \frac{1}{2} m_1(x_0, y_0)\dot{x}_0^2 + m_2(x_0, y_0)\dot{x}_0\dot{y}_0 + \frac{1}{2} m_3(x_0, y_0)\dot{y}_0^2 \\ &+ V(x_0, y_0) \ , \end{aligned} \tag{7.4.5}$$

we can monitor it as a function of time as a check on the numerical integrator. In Figure 7.10 we plot the Hamiltonian as a function of time corresponding to the variables plotted in Figure 7.9. From this plot we see that the Hamiltonian is indeed quite well conserved, although there is obviously something drastic happening when $x_0 = 0$. An even greater accuracy, up to about 1 part in 10^{-9} can be achieved by decreasing the error tolerances of the numerical integrator. The fact that the Hamiltonian is conserved so well, coupled with the fact that the code accurately reproduces the oscillation period, indicates that the equations have been coded properly and that the integrator is working. This leads us to consider what features of the ODEs could cause such behavior, or more importantly, to find the root of the problem in the original ansatz.

7.5 Limiting Analysis: $x_0 \to 0$

Since the problem occurs for small values of the x_0 coordinate, it is useful to examine the equations of motion in this limiting case. Using the results from Table 7.1, one arrives at the following equations after a bit of algebra:

$$\ddot{x}_0 \approx x_0\left[Dm^2y_0^2(\dot{x}_0^2-1)+\frac{C}{y_0^2}-2m^2\right]+O(x_0^2)\ , \tag{7.5.1}$$

$$x_0\ddot{y}_0 \approx -2\dot{x}_0\dot{y}_0+Ey_0^3x_0(\dot{x}_0^2-1)+O(x_0^2)\ , \tag{7.5.2}$$

with the constants C, D and E given by

$$C = \frac{\pi^2}{15}\approx 0.65 \tag{7.5.3}$$

$$D = \frac{2}{5}\left(\frac{4C-3}{4C-1}\right)\approx -0.09 \tag{7.5.4}$$

$$E = \frac{2}{5}(C-.25)^{-1}\approx 0.98\ . \tag{7.5.5}$$

Since these equations are valid only for small x_0, one cannot divide the equation for y_0 by x_0. The numerical integrator used to solve the equations of motion is an algebraic-differential equation solver, which means that the form for the y_0 equation given in Eq. (7.5.2) is what is used in the code. Even though the integrator can handle such a potential singularity, it is clear from Eq. (7.5.2) that we can expect some very rapid changes in the y_0 coordinate as $x_0 \to 0$.

Another interesting feature of the limiting equations is the appearance of the factors $\dot{x}_0^2 - 1$. Recall that the velocity in Figure 7.9 reached a minimum value of -1 before turning around. Since this behavior has been observed for initial velocities other than that shown in Figure 7.9, one is led to look for zeroes of the right-hand side of Eq. (7.5.1) (zeroes in $\ddot{x}_0$ correspond to "turning points" in $\dot{x}_0$). Such a turning point would occur for $\dot{x}_0^2 = 1$ if

$$\frac{C}{y_0^2} = 2m^2\ . \tag{7.5.6}$$

In the simulations we have taken $m = \lambda = 1$ (these choices were made so that direct comparisons could be made with the PDE simulations [15]), so Eq. (7.5.6) requires that $y_0 \approx 0.57$, a condition which does not hold in Figure 7.9 and in other simulations. Presently an effort is being made to search for zeroes of the right-hand side of the exact x_0 equation to see if a turning point for $\dot{x}_0^2 = 1$ is a generic feature.

From the limiting forms of the equations of motion, we see that there must be some very rapid behavior near $x_0 = 0$. This fact is further supported by an

examination of the Hamiltonian surface on which the motion must occur. The Hamiltonian surface is a three dimensional surface embedded in the four dimensional state space $(x_0, \dot{x}_0, y_0, \dot{y}_0)$. The most convenient method of examining such a surface is to fix one of the coordinates and examine a two dimensional section of the three dimensional surface. To compute values on this surface, it is easiest to solve for either $\dot{x}_0$ or $\dot{y}_0$ using the quadratic formula which yields

$$\dot{x}_0(x_0, y_0, \dot{y}_0; H_0) = -m_2\dot{y}_0 \pm \frac{1}{m_2}\sqrt{m_2^2\dot{y}_0^2 - 2m_1\left(\frac{1}{2}m_3\dot{y}_0^2 + V - H_0\right)} \,, \quad (7.5.7)$$

$$\dot{y}_0(x_0, y_0, \dot{x}_0; H_0) = \frac{-m_2}{m_3}\dot{x}_0 \pm \frac{1}{m_3}\sqrt{m_2^2\dot{x}_0^2 - 2m_3\left(\frac{1}{2}m_1\dot{x}_0^2 + V - H_0\right)} \,. \quad (7.5.8)$$

Using the limiting forms given in Table 7.1 we can compute the values that $\dot{y}_0$ must take as $x_0 \to 0$:

$$\dot{y}_0(x_0, y_0, \dot{x}_0; H_0) \to \frac{\alpha y_0}{x_0}\left[-\dot{x}_0 \pm \sqrt{\dot{x}_0^2 - \beta\left(\frac{m_1\dot{x}_0^2}{2} + V - H_0\right)}\right] \,, \quad (7.5.9)$$

with α and β given by

$$\alpha = \frac{15}{2\pi^2} \,, \quad (7.5.10)$$

$$\beta = \frac{\pi^2}{5\sqrt{2}m^3y_0} \,. \quad (7.5.11)$$

This limiting form for $\dot{y}_0$ tells us that for finite $\dot{x}_0$, unless

$$\frac{1}{2}m_1\dot{x}_0^2 + V - H_0 = 0 \,, \quad (7.5.12)$$

$\dot{y}_0$ will diverge as x_0 approaches 0. If Eq. (7.5.12) is satisfied, we find that $\dot{y}_0$ is zero. This type of behavior in $\dot{y}_0$ is confirmed in Figure 7.9. If we plot the values of $\dot{y}_0$ as computed from the exact quadratic formula given in Eq. (7.5.8) we find similar behavior. Figure 7.11 shows plots of these $\dot{y}_0$ values for fixed $\dot{x}_0$. Plots for different values of $\dot{x}_0$ have similar features. Since the coordinates must evolve such that they remain on this Hamiltonian surface (also verified by the plots of the Hamiltonian such as Figure 7.10), we must conclude that as x_0 approaches zero, the $\dot{y}_0$ parameter must take on very large values. This in turn causes y_0 to take on large values which corresponds to a very sharp kink, much sharper than is physically reasonable. Therefore it appears that the ansatz in Eq. (7.2.3) is not sufficient to capture the observed behavior. Two possible deficiences of the ansatz are that it does not include any radiation degrees of freedom and that it

Figure 7.11: A two-dimensional Hamiltonian section for fixed $\dot{x}_0 = -0.9$.

does not include relativistic terms. Since the actual simulations showed that for small initial velocities very little energy was carried via emission of radiation, it would seem that the relativistic corrections are at the root of the problem. This is further supported by the fact that in all of the simulations of the ODEs, the velocities attained were relativistic. In addition we can look at the values of $\dot{x}_0$ allowed by the Hamiltonian. In Figure 7.12 we plot the Hamiltonian section for fixed $y_0 = 1$. Here we see that all of the velocities for x_0 near zero are indeed close to -1. What is even more striking is the fact that the velocity never exceeds -1, but turns around just before the limiting velocity is attained.

7.6 A "Relativistic" Ansatz

Since the original Lagrangian is Lorentz invariant, the boosted kink

$$\tanh\Big[\frac{m(x+x_0)}{\sqrt{2}\sqrt{1-\dot{x}_0^2}}\Big] \ , \tag{7.6.1}$$

is a solution to the equations of motion. This prompts one to modify the ansatz given in Eq. (7.2.3) to include the relativistic "γ" factor

$$\Phi_A(x,t) = \frac{m}{\sqrt{\lambda}}\left\{1 - \tanh\Big[\frac{my_0\gamma(x-x_0)}{\sqrt{2}}\Big] + \tanh\Big[\frac{my_0\gamma(x+x_0)}{\sqrt{2}}\Big]\right\} \ , \tag{7.6.2}$$

with γ given by

$$\gamma \equiv \frac{1}{\sqrt{1-\dot{x}_0^2}} \ . \tag{7.6.3}$$

By including the factor of γ we ensure that the width of the kink and antikink will decrease as the velocity increases. This may in fact take the place of the y_0 parameter, however we shall keep both coordinates initially to ensure the greatest flexibility.

Again we insert this ansatz into the expression for the Lagrangian density and integrate over space to obtain an effective Lagrangian

$$\begin{aligned} & L(x_0,\dot{x}_0,\ddot{x}_0,y_0,\dot{y}_0) = \gamma^2\left[\frac{1}{2}\tilde{m}_1(x_0,y_0)\dot{x}_0^2 + \tilde{m}_2(x_0,y_0)\dot{x}_0\dot{y}_0 + \frac{1}{2}\tilde{m}_3(x_0,y_0)\dot{y}_0^2\right] \\ + \ & \frac{m^4\sqrt{2}}{2\lambda}\left\{\frac{4\gamma\dot{x}_0\ddot{x}_0}{m^3y_0^2}\Big[2\dot{y}_0 + \gamma^2 y_0\dot{x}_0\ddot{x}_0\Big]w_3(z_0) + \frac{2\gamma^3 x_0 y_0\dot{x}_0^2\ddot{x}_0}{m}w_3(z_0)\right\} \\ - \ & \gamma^2\tilde{V}_1(x_0,y_0) - \tilde{V}_2(x_0,y_0) \ , \end{aligned} \tag{7.6.4}$$

where

$$\tilde{m}_i \ \equiv m_i(x_0,\gamma y_0) \ , \tilde{V}_i \ \equiv V_i(x_0,\gamma y_0) \ , \tag{7.6.5}$$

Figure 7.12: A two dimensional Hamiltonian section for fixed $y_0 = 1$.

and the expressions for the functions $w_i(z_0)$ are given in Appendix I. Since this effective Lagrangian depends on the second time derivative of x_0, one must use second order variational methods [138, 139] to obtain the Euler-Lagrange equations of motions. Since the Lagrangian does not depend of the second time derivative of y_0, the standard Euler-Lagrange equation applies. The equation for x_0 is [140]

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}_0} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_0} + \frac{\partial L}{\partial x_0} = 0 \ . \tag{7.6.6}$$

Due to the additional terms in the Lagrangian and the equation of motion for x_0, the resulting set of coupled equations for x_0 and y_0 are extremely long and complex. Presently these equations are being derived and coded.

Although no results are yet available for the set of equations which result from carrying out the calculations in Eq. (7.6.6), the data in the previous sections indicates that a "relativistic" approach as outlined has some promise. The failure of the intuitive ansatz used in the previous sections indicates that either the relativistic corrections are needed or that a more sophisticated ansatz is required. Perhaps one needs to include the phonon degrees of freedom to achieve the quantitative agreement sought.

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