Chapter 8

Conclusions and Future Work

One of the most important results of this work is the formulation of a Newton's equation for the kink center of mass variable $X(t)$. The force which the kink experiences is found to depend on the phonons radiated by the interaction of the kink with the perturbation. The fact that these phonons appear in the kink center of mass equation demonstrates that the kink is an extended particle with internal degrees of freedom. This conclusion could be reached from purely numerical experiments in which the original PDE is solved "exactly". However the explicit separation of the degrees of freedom into kink and phonon components makes the analysis more physical. The first-order motion is especially easy to deduce since an effective potential exists for the kink center of mass variable. The second order motion is complicated by the appearance of the phonon degrees of freedom, but it is still tractable numerically. The specific applications of the perturbation theory presented here were chosen to mimic as closely as possible situations which might appear in real systems. One could easily imagine other perturbations which are less accurate approximations of the real situation (such as delta function potentials) for which the entire analysis (through second order) could be carried out analytically. This was illustrated when the thermal fluctuations were studied in Chapter 6. There we were able to derive a Fokker-Planck equation which again showed that the kink behaves, to lowest order, as a Newtonian particle.

One of the most interesting aspects of the present perturbation theory is the ability to describe shape changes of the kink waveform. In section 5.6 we saw that the ψ field accurately predicted the correct shape change for a kink which entered a new medium in which the limiting propagation speed was higher than the original medium. This shape change illustrates the fact that although the kink obeys Newtonian dynamics, it does not behave as a point particle; rather it behaves like an extended, deformable particle. The fact that the kink is an extended particle is not surprising, especially when one views the kink in the context of the pendulum chain. Here, we see that the kink is the result of a "cooperation" of many of the individual single pendulum degrees of freedom. The transformation which is the basis for our perturbation theory simply redistributes these degrees of freedom so that the kink may be described by only one coordinate.

In addition to describing kink shape changes, the ψ field must describe any phonons emitted and their influence on the kink motion. In section 5.3 we saw that one of the results of this interaction between the kink and phonon degrees of freedom is a transfer of energy from the kink to the phonons. This evidenced itself in a final kink velocity which was slightly lower than the initial velocity. In addition, we observed oscillations in the velocity about this final value, again indicative of a transfer of energy to and from other modes. One could imagine that similar energy transfer could occur when the kink-bearing system is coupled to different degrees of freedom. For example, in magnetic systems the kink represents a domain wall while the phonons represent spin waves. If the magneto-acoustic coupling constant is strong enough, one might find additional lattice vibrations induced when the kink collides with a magnetic impurity. This would possibly be observable as a contribution to the kink "viscosity".

Having gained some confidence with the method presented, we can look ahead to see other possible applications of the method. Due to the rather general form which the perturbation can take, many other relevant perturbations can be studied. It should be remembered that there are some interesting situations for which there is no perturbation present but in which the initial conditions are nontrivial. The simulations of Wada and Schrieffer [67] and Ogata and Wada [68] fall into this class, since they considered the collision of a prepared phonon packet with a stationary kink. To lowest order they find that the phonon packet is merely phase-shifted relative to the case in which no kink is present. To higher order they find the generation of reflected and transmitted phonons of frequency $2\omega_{\bar{q}}$ where $\bar{q}$ is the mean wave vector of the phonon packet. In addition the first and second order phase shifts experienced by the kink could be computed as a function of the mean frequency $\bar{q}$ and compared with the previous results. Through the use of the collective coordinate $X(t)$ one could hopefully come to a better understanding of the momentum transfer which occurs in these collisions.

Although the formal theory derived in Chapter 3 is set up to study time-dependent perturbations $v(x,t)$, our codes have not as yet been generalized to handle this situation. One of the interesting problems which could be studied with this capability is the damped, harmonically driven sine-Gordon equation. This particular equation has been the subject of several studies [28, 141, 142]. Although these simulations were carried out on the finite line, it would be interesting to see if the same types of chaos observed there arise in the infinite system. Since our ansatz includes only one kink and assumes that the phonon field ψ is small, the standard period-doubling route to chaos would evidence itself indirectly by the development of an instability. The instability would evidence itself by the developement of a phonon field which would try to produce another kink-like structure. Since this would require a rather large phonon field, this approach would only be able to indicate the onset of the period-doubled regime. The method presented by Tomboulis and Woo [46] may be better suited to study this system since it allows for more than one soliton component to be present. Even better suited to study this problem are the modulation equations derived by Erconali, McLaughlin and Forest [36] which are tailored to study the finite line with multiple solitons present. Currently Flesch and Forest are applying these equations to this problem. In particular they are trying to reproduce the behavior observed by Ariyasu and Bishop [143] in their simulations. In particular, Ariyasu and Bishop have observed an interesting hystersis in the damped driven sine-Gordon equation.

Another area which merits further study is our work involving the Fokker-Planck equation for the phase-space distribution function $P(X, p; t)$. In having derived the derivative transformation (see Appendix G) a major technical problem has been solved. It now remains to develop a convergent procedure which yields corrections to the first-order distribution functions already derived. In order to verify (or negate) the adiabatic assumption which allowed us to factor the phase-space distribution function into a product of a phonon and kink distribution function, the time dependence of solutions to the first-order Fokker-Planck equations must be investigated. Having resolved the question of equilbration times for the undriven system, a new steady state ansatz would be required to study the driven system. The resulting equations would allow us to calculate transport coefficients such as mobilities. One could attack the problem of transport from a more fundamental Boltzmann equation [144] approach. Once again the canonical nature of the transformation is of great benefit since the Jacobian of the transformation is unity.

Besides using the present theory to study additional applications, there is additional formal work to be done. As it stands, our theory is restricted to the study of low velocity kinks. Since the "Lorentz-boosted" solution

$$\phi_c\Big[\frac{x - vt}{\sqrt{1 - v^2}}\Big]$$

satisifies the unperturbed equation, one wonders if a canonical transformation is available which uses such a solution as a starting point. If such an approach fails, one might be able to make some progress using covariant collective coordinates [145, 146, 147].

A further relevant question involves the quantization of the system. One of the interesting problems to be attacked is soliton tunneling in the presence of pertubations. So far, however, only the statistical mechanics of the quantum system system has received attention [148, 149]. Tomboulis [45] approaches the problem semiclassically by promoting the variables to operators and the *Dirac* brackets to commutators, a procedure which is well defined because the transformation to the new variables is indeed canonical. A rather subtle point in carrying out this promotion involves using the correctly symmetrized form for the momentum operator Π_0. Once this promotion is completed one expands the ψ field in terms of normal-mode creation and annihilation operators. Transition matrix elements can then be calculated.

Gervais et al. [47] approach the quantization problem for the unperturbed system via a functional integral approach, writing the action in terms of the new variables. The point canonical transformation to the new variables in the action must be made carefully in order to be consistent [150]. When these points are taken care of, both the semiclassical and functional integral methods yield the same results to lowest order. Diagrammatic techniques based on the semiclassical approach [112] and functional integral [151] methods as applied to the perturbed problem are currently being investigated.

All of the quantum calculations mentioned above are carried out in the one soliton sector of the Fock space, that is to say, only one soliton is assumed to be present. Since many solitons can be present in a system, one really needs a formalism which can handle such instances. Of particular importance is the two-soliton case since the interaction between the solitons can greatly change the final state of the system as has been seen in the ϕ^4 kink-antikink collisions studied in Chapter 7. A rather natural approach would be to have creation and annihilation operators *for the solitons*. This approach has only been briefly studied by Mandelstam [152]. The collective-coordinate approach will undoubtedly be of value in these future investigations.

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[173] See Ref. 89 p. 967, formulae # 8.471.

[174] See Ref. 89 p. 666, formula # 6.511.6.

[175] See Ref. 89 p. 973, formula # 8.511.1.

[176] An error in Eq. 6 of §16.59 of Ref. 91 has been corrected by A. S. Yudina and P. I. Kuznetsov in USSR Comp. Math. and Math. Phys. **11**, 258 (1971).

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