# Appendix A

# Equations of Motion via Euler-Lagrange Formalism

In this appendix we derive the equations of motion for the dynamical variables and from these we derive second-order equations for $X(t)$ and $\psi(x,t)$. As mentioned in Chapter 2, this involves taking the Dirac bracket of the dynamical variables with the Hamiltonian. To this end, the Hamiltonian was written as the sum of three terms, an unperturbed term $H_0$, an interaction term $H_{int}$, and $H_{\psi_0}$ which involves terms proportional to the background field $\psi_0(x,t)$. The reason for writing $H$ in this particular form is that Tomboulis [45] has already computed the Dirac bracket of the dynamical variables with the unperturbed contribution to the Hamiltonian. The results of these calculations are [45]

$$\begin{aligned}
\{X, H_0\} &= \frac{p + \int \pi\psi'}{M_0(1+\xi/M_0)^2} \ , && \text{(A.1)}\\
\{p, H_0\} &= 0 \ , && \text{(A.2)}\\
\{\psi, H_0\} &= \pi(x,t) + \frac{p + \int \pi\psi'}{M_0(1+\xi/M_0)^2}\left[\psi'(x,t) - \frac{1}{M_0}\xi\phi_c'(x)\right] , && \text{(A.3)}\\
\{\pi, H_0\} &= \frac{p + \int \pi\psi'}{M_0(1+\xi/M_0)^2}\left[\pi'(x,t) + \frac{1}{M_0}\phi_c'(x)\int \pi\phi_c'' - \frac{p+\int\pi\psi'}{M_0(1+\xi/M_0)}\phi_c''(x)\right] \\
&\quad +\psi''(x,t) - V'(\psi,\phi_c) + \frac{1}{M_0}\phi_c'(x)\left(\int \psi'\phi_c'' + \int V'\phi_c'\right) . && \text{(A.4)}
\end{aligned}$$

Using the brackets given in Eqs. (3.3.10-11) one can calculate the following brackets

$$\begin{aligned}
\{X, H_{\psi_0} + H_{int}\} &= \frac{A(X,t)}{M_0(1+\xi/M_0)} \ , && \text{(A.5)}\\
\{p, H_{\psi_0} + H_{int}\} &= -\int \pi'(x-X,t)\dot{\psi}_0(x,t) + \int \phi_c''(x-X)\psi_0'(x,t)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{p+\int \pi\psi}{M_0(1+\xi/M_0)} \int \phi_c''(x-X)\dot{\psi}_0(x,t) \\
&+ \quad \int \psi''(x,t)\psi_0'(x+X,t) + \\
&+ \quad \int U'[\Phi(x,t)][\phi_c'(x-X)+\psi(x-X,t)] \\
&+ \quad \frac{\partial}{\partial X}\int v(x,t)F[\Phi(x,t),\Phi_x(x,t)]\ , \qquad \text{(A.6)}
\end{aligned}
$$

$$
\begin{aligned}
\{\psi(x,t), H_{\psi_0}+H_{int}\} &= -\dot{\psi}_0(x+X,t) + \frac{\phi_c'(x)A}{M_0} + \\
& \frac{A}{M_0(1+\xi/M_0)}\Big[\psi'(x,t) - \frac{\xi\phi_c'(x)}{M_0}\Big]\ , \qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
\{\pi(x,t), H_{\psi_0}+H_{int}\} &= \Big(1-\mathcal{P}_{\phi_c}\Big)\Big\{\frac{A\pi'(x,t)}{M_0(1+\xi/M_0)} - \frac{A(p+\int\pi\psi')}{M_0(1+\xi/M_0)^2}\phi_c'' \\
+\psi_0''(x+X,t) &- (\Delta U)' + v(x+X,t)F_{10}[\Phi(x+X,t),\Phi_x(x+X,t)] \\
&+ \frac{d}{dx}\Big[v(x+X,t)F_{10}[\Phi(x+X,t),\Phi_x(x+X,t)]\Big]\Big\}\ , \qquad \text{(A.8)}
\end{aligned}
$$

where

$$
A(X,t) = \int \phi_c'(x-X)\dot{\psi}_0(x,t)\ , \qquad \text{(A.9)}
$$

$$
(\Delta U)' = \frac{\partial}{\partial\Phi}\Big(\Delta U[\Phi]\Big)\Big|_{\Phi=\Phi(x+X,t)}\ , \qquad \text{(A.10)}
$$

and primes and dots denote derivatives with respect to the first and second arguments respectively (dots are not total time derivatives). Combining Eqs. (A1) to (A8) we can write

$$
\begin{aligned}
\dot{X} &= \frac{p+\int\pi\psi'}{M_0(1+\xi/M_0)^2} + \frac{A(X,t)}{M_0(1+\xi/M_0)}\ , \qquad \text{(A.11)} \\
\dot{p} &= \frac{\partial}{\partial X}\int v(x,t)F[\Phi,\Phi_x] - \int \pi'(x-X,t)\dot{\psi}_0(x,t) \\
&+ \frac{p+\int\pi\psi'}{M_0(1+\xi/M_0)}\int\phi_c''(x-X)\dot{\psi}_0(x,t) \\
&+ \int\phi_c''(x-X,t)\psi_0(x',t) + \int\psi''(x-X)\psi_0'(x,t) \\
&+ \int U'[\Phi(x,t)]\Big(\phi_c'(x-X)+\psi(x-X,t)'\Big)\ , \qquad \text{(A.12)} \\
\dot{\psi}(x,t) &= \pi(x,t) + \frac{p+\int\pi\psi'}{M_0(1+\xi/M_0)^2}\Big[\psi'(x,t) - \frac{\xi\phi_c'}{M_0}\Big] - \dot{\psi}_0(x+X,t) \\
&+\frac{\phi_c'(x)A(X,t)}{M_0} + \frac{A(X,t)}{M_0(1+\xi/M_0)}\Big[\psi'(x,t) - \frac{\xi\phi_c'(x)}{M_0}\Big]\ , \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\pi}(x,t) &= \Big(1-\mathcal{P}_{\phi_c}\Big)\Bigg\{\frac{p+\int \pi\psi'}{M_0(1+\xi/M_0)^2}\Big[\pi'(x,t)-\phi_c''(x)\frac{p+\int\pi\psi'}{M_0(1+\xi/M_0)}\Big] \\
&+ \psi''(x,t)-V'(\psi,\phi_c)+\frac{A\pi'(x,t)}{M_0(1+\xi/M_0)}-\frac{A(p+\int\pi\psi')}{M_0^2(1+\xi/M_0)^2}\phi_c'' \\
&+ \psi_0''(x+X,t)-(\Delta U)'+v(x+X,t)F_{10}[\Phi(x+X,t),\Phi_x(x+X,t)] \\
&- \frac{d}{dx}\Big[v(x+X,t)F_{10}[\Phi(x+X,t),\Phi_x(x+X,t)]\Big]\Bigg\}\ . \qquad \text{(A.14)}
\end{aligned}
$$

Next we derive a second-order equation for the kink center of mass variable $X(t)$. We begin by taking a total time derivative of Eq. (A11) which may be written

$$
\begin{aligned}
\ddot{X} = \frac{\dot{p}+\int\dot{\pi}\psi'+\int\pi\dot{\psi}'}{M_0(1+\xi/M_0)^2}-\frac{\int\phi_c'\dot{\psi}'}{M_0(1+\xi/M_0)}\dot{X} \ &- \ \frac{p+\int\pi\psi'}{M_0^2(1+\xi/M_0)^2}\int\phi_c'\dot{\psi}' \\
&+ \ \frac{1}{M_0(1+\xi/M_0)}\frac{d}{dt}A\ . \qquad \text{(A.15)}
\end{aligned}
$$

We consider each of the terms in Eq. (A.15) in turn, first treating the $\int\dot{\pi}\psi'$ term. Using Eq. (A.14) for $\dot{\pi}$ and collecting terms we have

$$
\begin{aligned}
\int\dot{\pi}\psi' &= \dot{X}\int\pi'\psi'+\frac{\dot{X}\xi}{M_0}\int\phi''\pi-\frac{(p+\int\psi'\pi)^2}{M_0(1+\xi/M_0)^3}\int\phi_c''\psi' \\
&- \int\Big[U'(\phi_c+\psi)-U'(\phi_c)\Big]\psi'+\frac{\xi}{M_0}\int\phi_c'[U'(\phi_c+\psi)-U'(\phi_c')] \\
&- \frac{A(p+\int\pi\psi')}{M_0^2(1+\xi/M_0)^2}\int\phi_c''\psi'+\int\psi_0''(x+X,t)\psi'-\frac{\xi}{M_0}\int\psi_0''(x+X,t)\phi_c' \\
&- \int(\Delta U)'\psi'+\frac{\xi}{M_0}\int(\Delta U)'\phi_c'+\frac{\xi}{M_0}\int\psi'\phi'' \\
&+ \int v(x,t)\psi'(x-X,t)F_{10}-\int\psi'(x-X,t)\frac{d}{dx}\Big(v(x,t)F_{01}\Big) \\
&- \frac{\xi}{M_0}\Big[\int v(x,t)\phi_c'(x-X,t)F_{10}-\int\phi_c'(x-X,t)\frac{d}{dx}\Big(v(x,t)F_{01}\Big)\Big] \qquad \text{(A.16)}
\end{aligned}
$$

Next we collect four of the terms in Eq. (A.16) together and write

$$
\begin{aligned}
&- \int\Big[U'(\phi_c+\psi)-U'(\phi_c)\Big]\psi'+\frac{\xi}{M_0}\int\phi_c'[U'(\phi_c+\psi)-U'(\phi_c)] \\
&\qquad -\int(\Delta U)'\psi'+\frac{\xi}{M_0}\int(\Delta U)'\phi_c' = \\
&(1+\frac{\xi}{M_0})\int U'[\Phi(x+X,t)]\phi_c'+\int\phi_c''\psi'+\int U'[\Phi(x,t)]\ \psi_0'(x+X,t)\ . \qquad \text{(A.17)}
\end{aligned}
$$

This allows us to write

$$
\begin{aligned}
\int \dot{\pi}\psi' &= \dot{X}\int \pi'\psi' + \frac{\dot{X}\xi}{M_0}\int \phi_c''\pi - \frac{(p+\int \pi\psi')^2}{M_0(1+\xi/M_0)^3}\int \phi_c''\psi' \\
&+ (1+\frac{\xi}{M_0})\int U'[\Phi(x,t)]\phi_c'(x-X) + \int \phi_c''\psi' \\
&- \int U'[\Phi(x+X,t)][\phi_c'(x,t)+\psi'(x,t)] + \frac{\xi}{M_0}\int \psi'\phi_c'' \\
&- \frac{A(p+\int \pi\psi')}{M_0^2(1+\xi/M_0)^2}\int \phi_c''\psi' + \int \psi_0''(x+X,t)\psi' - \frac{\xi}{M_0}\int \psi_0''(x+X,t)\phi_c' \\
&+ \int v(x,t)\psi'(x-X,t)F_{10} - \int \psi'(x-X,t)\frac{d}{dx}\Big(v(x,t)F_{01}\Big) \\
&- \left[\frac{\xi}{M_0}\int v(x,t)\phi_c'(x-X,t)F_{10} - \int \phi_c'(x-X,t)\frac{d}{dx}\Big(v(x,t)F_{01}\Big)\right] \quad \text{(A.18)}
\end{aligned}
$$

Next we we consider the $\int \pi\dot{\psi}'$ term for which we can write

$$
\int \pi\dot{\psi}' = \dot{X}\int \pi\psi'' - \frac{\xi\dot{X}}{M_0}\int \pi\phi_c'' - \int \pi\dot{\psi}_0(x+X,t) + \frac{A}{M_0}\int \pi\phi_c'' \ . \quad \text{(A.19)}
$$

Combining Eqs. (A.18) and (A.19) and using Eq. (A.12) for $\dot{p}$ we can write for the numerator of the first term of Eq. (A.15),

$$
\begin{aligned}
\dot{p} + \int \dot{\pi}\psi' &+ \int \pi\dot{\psi}' = -\Big(1+\frac{\xi}{M_0}\Big)\int v(x,t)\Big[\phi_c'(x-X)F_{10} + \phi_c''(x-X)F_{01}\Big] \\
&+ \frac{p+\int \pi\psi'}{M_0(1+\xi/M_0)}\int \phi_c''(x-X)\dot{\psi}_0(x,t) \\
&+ (1+\frac{\xi}{M_0})\Bigg[(1-\dot{X}^2)\int \psi'\phi_c'' \\
&- \int \psi_0''(x,t)\phi_c'(x-X) + \int U'[\Phi(x,t)]\phi_c'(x-X)\Bigg] \\
&+ \frac{A(X,t)(p+\int \pi\psi')}{M_0^2(1+\xi/M_0)^2}\int \phi_c''\psi' + \frac{A^2(X,t)}{M_0^2(1+\xi/M_0)}\int \phi_c''\psi' \\
&+ \frac{A(X,t)}{M_0}\int \pi\phi_c'' \ . \quad \text{(A.20)}
\end{aligned}
$$

Lastly, we compute $\int \phi_c'\dot{\psi}'$ and $dA/dt$:

$$
\int \phi_c'\dot{\psi}' = \int \phi_c'\pi + \dot{X}\int \phi_c'\psi'' - \int \phi_c'\dot{\psi}_0'(x+X,t) \quad \text{(A.21)}
$$

$$
\frac{dA(X,t)}{dt} = \int \ddot{\psi}_0(x+X,t)\phi_c'(x) + \dot{X}\int \dot{\psi}_0'(x+X,t)\phi_c'(x) \ . \quad \text{(A.22)}
$$

Combining Eqs. (A.20-22) we finally have

$$
\begin{aligned}
\ddot{X} &= \frac{1}{M_0(1+\xi/M_0)}\Big\{ -\int v(x,t)\Big[\phi_c'(x-X)F_{10}[\Phi(x,t),\Phi_x(x,t)] \\
&\qquad\qquad +\phi_c''(x-X)F_{01}[\Phi(x,t),\Phi_x(x,t)]\Big] \\
&+ \int(\ddot{\psi}_0-\psi_0'')\phi_c'(x-X)+\int U'[\Phi(x,t)]\phi_c'(x-X)+(1+\dot{X}^2)\int\psi'\phi_c'' \\
&\qquad\qquad -2\dot{X}\int\pi'\phi_c'+2\dot{X}\int\phi_c'(x)\dot{\psi}_0'(x+X,t)\Big\}\ , \qquad \text{(A.23)}
\end{aligned}
$$

where we have repeatedly made use of Eq. (A.11). One can use the $\psi_0$ equation to replace the $\ddot{\psi}_0-\psi_0''$ term if desired.

In the same manner we could derive an exact second-order equation for $\psi(x,t)$. However, it would be extremely long and would not give us as much insight as the exact second-order equation for $X(t)$. Rather, we will derive a second-order differential equation for $\psi(x,t)$ which is valid to first-order in the perturbation strength. Taking the total time derivative of $\dot{\psi}$ given in Eq. (A.13) we have, keeping only terms of first-order in the perturbation,

$$
\ddot{\psi}(x,t)=\dot{\pi}(x,t)-\ddot{\psi}_0(x+X,t)+\frac{\phi_c'(x)}{M_0}\frac{dA(X,t)}{dt}\ . \qquad \text{(A.24)}
$$

Using expressions for $\dot{\pi}$ and $dA/dt$ given by Eqs. (A.14) and (A.22) we have, again keeping only first-order terms in $\lambda$,

$$
\begin{aligned}
\ddot{\psi}(x,t) &= \psi''(x,t)-V'(\psi,\phi_c)-\frac{\phi_c'(x)}{M_0}\Big[\int\psi\phi_c''+\int V'(\psi,\phi_c)\phi_c\Big] \\
&+ (1-\mathcal{P}_{\phi_c})\Big\{\psi_0''(x+X,t)-(\Delta U)'+v(x+X,t)F_{10}[\Phi(x+X,t),\Phi_x(x+X,t)] \\
&- \frac{d}{dx}\Big(v(x+X,t)F_{01}[\Phi(x+X,t),\Phi_x(x+X,t)]\Big)-\ddot{\psi}_0(x+X,t)\Big\}\ . \qquad \text{(A.25)}
\end{aligned}
$$

Next we use the facts that

$$
V'(\psi,\phi_c)=\psi(x,t)U''(\phi_c(x))+O(\lambda^2)\ , \qquad \text{(A.26)}
$$

$$
(\Delta U)'=\psi_0(x+X,t)U''(\phi_c)+O(\lambda^2)\ , \qquad \text{(A.27)}
$$

and

$$
\int\psi'\Big(\phi_c''+V'(\psi,\phi_c)\Big)=O(\lambda^2) \qquad \text{(A.28)}
$$

to write

$$
\begin{aligned}
&\ddot{\psi}(x,t) - \psi''(x,t) + \psi(x,t)U''(\phi_c) = \\
&\quad (1-\mathcal{P}_{\phi_c})\Bigg\{ -\ddot{\psi}_0(x+X,t) + \psi_0''(x+X,t) - \psi_0(x+X,t)U''(\phi_c) \\
&\qquad\qquad +v(x+X,t)F_{10}[\Phi(x+X,t),\Phi_x(x+X,t)] \\
&\qquad\qquad -\frac{d}{dx}\Big(v(x+X,t)F_{01}[\Phi(x+X,t),\Phi_x(x+X,t)]\Big)\Bigg\} \ . \qquad \text{(A.29)}
\end{aligned}
$$

Finally we use the fact that $\psi_0$ satisifies Eq. (3.3.8) to obtain

$$
\begin{aligned}
&\ddot{\psi}(x,t) - \psi''(x,t) + \psi(x,t)U''(\phi_c) = \\
&\quad (1-\mathcal{P}_{\phi_c})\Bigg\{ \Big[1-U''(\phi_c)\Big]\psi_0(x+X,t) + v(x+X,t)\Big[F_{10}[\phi_c,\phi_c'] - F_{10}[0,0]\Big] \\
&- \quad \frac{d}{dx}\Big[v(x+X,t)\Big(F_{01}[\phi_c,\phi_c'] - F_{01}[0,0]\Big)\Big]\Bigg\} \ . \qquad \text{(A.30)}
\end{aligned}
$$

# Appendix B

# Equations of Motion via Direct Substitution

In this appendix, we derive the equations of motion by simply substituting the kink variables for the original field variables in the equation of motion for the original fields. The Euler-Lagrange equation of motion for the original field $\Phi(x,t)$ can be derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\Phi_t^2 - \frac{1}{2}\Phi_x^2 - U(\Phi) + v(x,t)F[\Phi(x,t),\Phi_x(x,t)]\ . \tag{B.1}$$

We add to this equation of motion a phenomenological damping term of the form $\epsilon\dot{\Phi}(x,t)$ (when appropriate) thereby obtaining

$$\Phi_{tt} + \epsilon\Phi_t(x,t) - \Phi_{xx} + U'(\Phi) + \mathcal{G} = 0\ . \tag{B.2}$$

where $\mathcal{G}$ is given by

$$\mathcal{G} = \frac{d}{dx}\Big[v(x,t)F_{01}[\Phi(x,t),\Phi_x(x,t)]\Big] - v(x,t)F_{10}[\Phi(x,t),\Phi_x(x,t)]\ . \tag{B.3}$$

Using as an ansatz for $\Phi$

$$\Phi(x,t) = \phi_c[x - X(t)] + \psi[x - X(t),t] + \psi_0(x,t)\ , \tag{B.4}$$

we may compute the appropriate derivatives that occur in Eq. (B.2);

$$\begin{aligned}
\frac{d}{dt}\Phi(x,t) &= -\dot{X}\phi_c'[x - X(t)] - \dot{X}\psi'[x - X(t),t] + \dot{\psi}[x - X(t),t] \\
&+ \dot{\psi}_0(x,t)\ , \qquad \text{(B.5)} \\
\frac{d^2}{dt^2}\Phi(x,t) &= -\ddot{X}\Big\{\phi_c'[x - X(t)] + \psi'[x - X(t),t]\Big\} + \dot{X}^2\Big\{\phi_c''[x - X(t)]
\end{aligned}$$

$$
\begin{aligned}
& + \quad \psi''[x - X(t), t]\Big\} - 2\dot{X}\dot{\psi}'[x - X(t), t] \\
& + \quad \ddot{\psi}[x - X(t), t] + \ddot{\psi}_0(x, t) \ , \qquad \text{(B.6)} \\
\frac{d^2}{dx^2}\Phi(x, t) \quad &= \quad \phi_c''[x - X(t)] + \psi''[x - X(t), t] + \psi_0''(x, t) \ , \qquad \text{(B.7)}
\end{aligned}
$$

where as before, primes and dots represent partial derivatives with respect to the first and second arguments respectively (a dot does not represent a total time derivative). To obtain an equation of motion for the center of mass coordinate $X(t)$, we substitute the expressions in Eqs. (B.5-7) into Eq. (B.2) , multiply by $\phi_c'[x - X(t)]$ and integrate over $x$. Using Eqs. (3.3.6-7) we have

$$
\begin{aligned}
[M_0 + \xi]\ddot{X} \quad &= \quad \dot{X}^2 \int \psi''\phi_c' - 2\dot{X} \int \dot{\psi}'\phi_c' + \int [\ddot{\psi}_0(x, t) - \psi_0''(x, t)]\phi_c'(x - X) \\
&- \quad \int \psi''\phi_c' + \int U'[\Phi]\phi_c'(x - X) - \epsilon\dot{X}[M_0 + \xi] \\
& +\epsilon \int \phi_c'(x - X)\dot{\psi}_0(x, t) + \int \phi_c'(x - X)\mathcal{G} \ , \qquad \text{(B.8)}
\end{aligned}
$$

where we have also used the fact that

$$
\int \ddot{\psi}\phi_c' = \int \dot{\psi}\phi_c' = 0 \ . \qquad \text{(B.9)}
$$

To put Eq. (B.8) into a form more easily compared with Eq. (A.23), we use Eq. (A.13) to substitute for $\dot{\psi}$, which after collecting like terms gives us

$$
\begin{aligned}
\ddot{X} \quad = \quad \frac{1}{M_0(1 + \xi/M_0)} \Big\{ & - \int \phi_c'(x - X)\mathcal{G} + \int (\ddot{\psi}_0 - \psi_0'')\phi_c'(x - X) \\
& + \int U'[\Phi(x, t)]\phi_c'(x - X) + (1 + \dot{X}^2) \int \psi'\phi_c'' \\
& -2\dot{X} \int \pi'\phi_c' + 2\dot{X} \int \phi_c'(x)\dot{\psi}_0'(x + X, t) \\
& +\epsilon \int \phi_c'(x - X)\dot{\psi}_0(x, t) \Big\} - \epsilon\dot{X} \ , \qquad \text{(B.10)}
\end{aligned}
$$

which agrees with Eq (A.23) for $\epsilon = 0$.

Similarly we can derive the $\psi$ equation. To do this we merely write the full field equation in terms of the new variables which after rearrangement yields

$$
\begin{aligned}
& \ddot{\psi}[x - X(t), t] - \psi''[x - X(t), t] + U'[\Phi(x, t)] = \\
& \qquad \ddot{X}\Big\{\phi_c'[x - X(t)] + \psi'[x - X(t)]\Big\} - \dot{X}^2\Big\{\phi_c''[x - X(t)] + \psi''[x + X(t), t]\Big\} \\
& \quad + \quad 2\dot{X}\dot{\psi}'[x - X(t), t] - \ddot{\psi}_0(x, t) + \phi_c''[x - X(t)] + \psi_0''(x, t) - \mathcal{G} \\
& \quad - \quad \epsilon\Big[\dot{X}\Big(\phi_c'(x - X) + \psi'(x - X, t) + \dot{\psi}(x_X, t) + \psi_0(x_X, t)\Big)\Big] \ . \qquad \text{(B.11)}
\end{aligned}
$$

Before we carry out a perturbation expansion of Eq. (B.10) we transform to a frame which moves with the kink. Since we don't know $X(t)$ exactly, we can transform to a frame $[y = x - X^{(1)}(t)]$ whose origin moves according to the first-order kink motion. In this frame, the kink velocity is of second-order in the perturbation and therefore we can neglect all terms in Eq. (B.11) which are proportional to $\dot{X}$ leaving us with

$$\begin{aligned}
\ddot{\psi}[y,t] &- \psi''[y,t] + U'[\phi_c(y,t)] + \Big(\psi(y,t) + \psi_0[y + X^{(1)}(t)]\Big)U''[\phi_c(y)] = \\
&- \left\{\frac{\phi_c'(y)}{M_0}\int \phi_c'[y - X^{(1)}(t)]\mathcal{G}\int(\ddot{\psi}_0 - \psi_0'')\phi_c'[y - X^{(1)}(t)]\right\} \\
&- \ddot{\psi}_0(y,t) + \phi_c''[y - X^{(1)}(t)] + \psi_0''(y,t) + \mathcal{G} \\
&- \epsilon\Big[\dot{X}\Big(\phi_c'(x - X) + \psi'(x - X, t) + \dot{\psi}(x_X, t) + \psi_0(x_X, t)\Big)\Big] \\
&+ \epsilon\frac{\phi_c'(y)}{M_0}\int \phi_c'(x - x)\dot{\psi}_0(x,t)\ . \qquad \text{(B.12)}
\end{aligned}$$

Cancelling common terms and using the projection operator notation we have

$$\begin{aligned}
\ddot{\psi}[y,t] &- \psi''[y,t] + \psi(y,t)U''[\phi_c(y)] \\
&= \Big(1 - \mathcal{P}_{\phi_c}\Big)\Big\{-\mathcal{G} - \ddot{\psi}_0 + \psi_0'' - \psi_0[y + X^{(1)}(t)]U''[\phi_c(x)] \\
&- \epsilon\dot{\psi} + \dot{\psi}_0(y + X^{(1)})\Big\}\ . \qquad \text{(B.13)}
\end{aligned}$$

Finally using Eq. (3.3.8) to first-order in the perturbation strength, we have

$$\begin{aligned}
&\ddot{\psi}[y,t] - \psi''[y,t] + \psi(y,t)U''[\phi_c(y)] = \\
&\qquad \Big(1 - \mathcal{P}_{\phi_c}\Big)\Big\{\psi_0[y + X^{(1)}(t)]U''[\phi_c(x)] \\
&+ \ v[y + X^{(1)}(t)]\Big(F_{10}[\phi_c(y)], \phi_c'(y)] - F_{10}[0,0]\Big) \\
&- \ v'[y + X^{(1)}(t)]\Big(F_{01}[\phi_c(y)], \phi_c'(y)] - F_{01}[0,0]\Big) \\
&- \ v[y + X^{(1)}(t)]\Big(\phi_c'(y)F_{11}[\phi_c(y)], \phi_c'(y)] - \phi_c''(y)F_{02}[\phi_c(y)], \phi_c'(y)]\Big) \\
&- \ \epsilon\dot{\psi} + \dot{\psi}_0(y + X^{(1)})\Big\}\ , \qquad \text{(B.14)}
\end{aligned}$$

which is equivalent to what is given in Eq. (A.30).

# Appendix C

# Evaluation of the integral $J(\beta^2)$

The integral $J(\beta^2)$ [ Eq. (4.1.50)] differs from Hardy's integral for Lommel functions [91, 92] only in that in the denominator, $t^2+1$ is replaced by $t^2+\beta^2$. The only restriction placed on $\beta$ is that $\Re(\beta)>0$. We first consider the case in which $b<0$ for which we have from the tables [153],

$$J(\beta^2) = \frac{1}{\pi}\int_0^\infty \frac{t\ dt}{t^2+\beta^2}\sin\Big[at+\frac{b}{t}\Big] = \frac{1}{2}e^{-(a\beta_- - \frac{b}{\beta_-})}\ , \qquad \text{(C.1)}$$

where the restriction $\Re(b)>0$ is required.

For $b>0$ we distinguish between $b<a$ and $b>a$. The latter may be reduced to the $b<a$ case by using the relation [154],

$$\frac{1}{\pi}\int_0^\infty \frac{t\ dt}{t^2+\beta^2}\sin\Big[at+\frac{b}{t}\Big] = J_0(2\sqrt{ab}) - \frac{1}{\pi}\int_0^\infty \frac{t\ dt}{t^2+\frac{1}{\beta^2}}\sin\Big[\frac{a}{t}+bt\Big]\ . \qquad \text{(C.2)}$$

Therefore we need only consider $b<a$. Without loss of generality we may confine our attention to $|\beta|=1$ by writing $\beta=|\beta|e^{i\varphi}$ which allows us to write

$$\begin{aligned} J(\beta^2) &= \frac{1}{\pi}\int_0^\infty \frac{t\ dt}{|\beta|^2[\frac{t^2}{|\beta|^2}+e^{2i\varphi}]}\sin\Big[at+\frac{b}{t}\Big]\ , && \text{(C.3)} \\ &= \frac{1}{\pi}\int_0^\infty \frac{t\ dt}{t^2+e^{2i\varphi}}\sin\Big[a't+\frac{b'}{t}\Big]\ , && \text{(C.4)} \end{aligned}$$

where $a'$ and $b'$ are $a$ and $b$ scaled by $1/|\beta|$. Therefore, with $b<a$ and $|\beta|=1$, we define

$$x \equiv 2\sqrt{ab} \quad , \quad c \equiv \frac{1}{\beta}\sqrt{\frac{a}{b}}\ , \qquad \text{(C.5)}$$

in terms of which we may write $J(\beta^2)$ as

$$
\begin{aligned}
J(\beta^2) &= \frac{1}{\pi}\int_0^\infty \frac{t\ dt}{t^2+\beta^2}\sin\Big[\frac{x}{2}\Big(t\sqrt{\frac{a}{b}}+\frac{1}{t}\sqrt{\frac{b}{a}}\Big)\Big] \ , && \text{(C.6)}\\
&= \frac{c}{\pi}\int_{-\infty}^{\infty} \frac{e^u du}{ce^u+\frac{1}{ce^u}}\sin[x\cosh(u)] \ , && \text{(C.7)}\\
&= \frac{c}{\pi}\int_0^\infty du\Big\{\frac{e^{-u}}{ce^{-u}+(ce^{-u})^{-1}}+\frac{e^u}{ce^u+(ce^u)^{-1}}\Big\}\sin[x\cosh(u)] \ , && \text{(C.8)}\\
&= \frac{1}{2\pi}\int_1^\infty \frac{d\tau}{\sqrt{\tau^2-1}}\frac{c^2-1+2\tau^2}{\theta^2+\tau^2}\sin(x\tau) \ , && \text{(C.9)}
\end{aligned}
$$

with

$$
\theta \equiv \frac{1}{2}(c-\frac{1}{c}) = \frac{c'^2+1}{2c'}\Big\{\frac{c'^2-1}{c'^2+1}\Re(\beta)-i\Im(\beta)\Big\} \ , \tag{C.10}
$$

$$
c' \equiv \sqrt{\frac{b}{a}} \ . \tag{C.11}
$$

Since $\Re(b) > 0$ and $c' < 1$, $\theta$ is never pure imaginary, therefore $\theta^2$ does not lie on the negative real axis and the only poles of the integrand in Eq. (C.9) are at $\tau = \pm 1$. We evaluate Eq. (C.9) by considering the contour integral $\Gamma(\beta^2)$ given by

$$
\Gamma(\beta^2) \equiv \int_\Gamma \frac{dz\ e^{ixz}}{\sqrt{z^2-1}}\frac{c^2-1+2z^2}{\theta^2+z^2} \ . \tag{C.12}
$$

With the branch cuts chosen as in Figure C.1, $\Gamma(\beta^2)$ becomes

$$
\Gamma(\beta^2) = 2i\int_1^\infty \frac{d\tau\sin(x\tau)}{\sqrt{\tau^2-1}}\frac{c^2-1+2\tau^2}{\theta^2+\tau^2} - 2i\int_{-1}^{1}\frac{d\tau\ e^{(ix\tau)}}{\sqrt{1-\tau^2}}\frac{c^2-1+2\tau^2}{\theta^2+\tau^2} \ , \tag{C.13}
$$

therefore we have for $J(\beta^2)$,

$$
\begin{aligned}
J(\beta^2_-) &= \frac{1}{2\pi i}\frac{\Gamma(\beta^2)}{2}+\frac{1}{2\pi}\int_0^1\frac{d\tau\cos(x\tau)}{\sqrt{1-\tau^2}}\frac{c^2-1+2\tau^2}{\theta^2+z^2} \ , && \text{(C.14)}\\
&= \frac{Res[f(z);-i\theta]}{2}+\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}d\varphi\ \cos[x\cos(\varphi)]\frac{c^2-1+2\cos^2(\varphi)}{\theta^2+\cos^2(\varphi)} \ , && \text{(C.15)}
\end{aligned}
$$

where $Res[f(z);-i\theta]$ is the residue of $f(z)$ evaluated at $-i\theta$ with $f(z)$ given by the integrand of Eq. (C.12). In writing Eq. (C.14) we have used the fact the

Figure C.1: Contour for the integral $\Gamma(\beta^2)$

contributions to $\Gamma(\beta^2)$ from the large and small semicircles vanish when $R \to \infty$ and $\delta \to\ 0$ respectively. Evaluating the residue at the simple pole $-i\theta$ we have

$$Res[f(z); -i\theta] = e^{-(a\beta - \frac{b}{\beta})}\ . \tag{C.16}$$

The remaining integral in Eq. (C.15) may be evaluated by noting that

$$\frac{c^2 - 1 + 2\cos^2(\varphi)}{\theta^2 + \cos^2(\varphi)} = -4\sum_{k=1}^{\infty}(ic)^{2k}\cos(2k\varphi)\ . \tag{C.17}$$

Since $c < 1$, the sum in Eq. (C.17) is uniformly convergent and we may insert it into Eq. (C.15) and integrate term by term. We also make the substitution

$$\cos[x\cos(\varphi)] = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)\cos(2n\varphi)\ , \tag{C.18}$$

The double sum resulting from substitution of Eqs. (C.17) and (C.18) into (C.15) is reduced to a single sum by orthogonality of $\cos(2n\varphi)$ on $[0, \frac{\pi}{2}]$ , leaving us with

$$\frac{1}{2\pi}\int_0^{\infty}\frac{d\tau\cos(x\tau)}{\sqrt{1-\tau^2}}\frac{c^2 - 1 + 2\tau^2}{\theta^2 + z^2} = -\sum_{k=1}^{\infty}c^{2k}J_{2k}(x)\ , \tag{C.19}$$

$$= -\Lambda_2\Big[\frac{2b}{\beta}, 2\sqrt{ab}\Big]\ . \tag{C.20}$$

Finally collecting Eqs. (C.1), (C.16) and (C.20) we have

$$J(\beta^2) = \frac{1}{2}e^{-(a\beta-\frac{b}{\beta})} - \theta(b)\Lambda_2\Big[\frac{2b}{\beta}, \sqrt{2ab}\Big] \ , \tag{C.21}$$

where $\theta(b)$ is the Heaviside step function.

# Appendix D

# Lommel Functions of Two Variables

In Chapter 4 we obtained analytic expressions for several Green functions in terms of "modified" Lommel functions of two variables (not to be confused with Lommel functions of one variable $s_{\mu,\nu}(z)$ or $S_{\mu,\nu}(z)$). Since these functions are somewhat obscure, we shall review some of the existing literature which illustrates properties of Lommel functions and numerical techniques which have been applied to evaluate the functions. We end this appendix with the derivation of some properties of the modified functions which were useful in the derivation of the Green functions in Chapter 4.

Lommel functions were first studied by Lommel in his studies of diffraction at a straight edge [155] and a circular aperature [156]. In these works, Lommel gives a detailed discussion of what have become known as Lommel functions of two variables $U_n(w,s)$ and $V_n(w,s)$, which are defined by the series

$$U_n(w,s) = \sum_{m=0}^{\infty} (-1)^m \Big(\frac{w}{s}\Big)^{2m+n} J_{2m+n}(s) \ , \tag{D.1}$$

$$V_n(w,s) = \sum_{m=0}^{\infty} (-1)^m \Big(\frac{w}{s}\Big)^{-(2m+n)} J_{-(2m+n)}(s) \ . \tag{D.2}$$

Among the properties examined are recurrence relations, integral expressions and values for special arguments. He also gives some short tables and plots for real values of $w$ and $s$. Many of these results are reproduced in Appendix II of Walker's *The Analytical Theory of Light* [157] as well as in Watson [91]. Lommel functions continue to be of use in the study of the propagation of electromagnetic radiation in a variety of media [158, 159, 160, 161].

Although Lommel functions of two variables have not received a great deal of attention since Lommel's original publications, several new properties and/or relations have been discovered. In particular, Shastri [162] presents some integral transforms related to the Laplace transform of the Lommel functions along with integrals of Lommel functions multiplied by Bessel functions, Fresnel functions, sines and cosines (these are not Fourier transforms due to the scaled arguments used) and other Lommel functions. Dekanosidze [163] shows that the functions $U_n(\xi, \eta)$ are solutions of the hyperbolic equation

$$\frac{\partial^2 U}{\partial\xi\partial\eta} + U = 0 \ , \tag{D.3}$$

with $\xi = s^2/2w$ and $\eta = w/2$. In addition, he derives about 15 complicated relations between the functions, one of the simpler being

$$U_\nu(w,s) = U_\nu(s^2/w, s) + \frac{1}{2}\int\limits_{s^2/w}^{w} dx\ J_{\nu-1}(x) J_0\Big[\sqrt{\big(x - s^2/w\big)(x-w)}\Big] \ . \tag{D.4}$$

Several of these relations may prove useful when the functions are actually evaluated numerically.

The first numerical evaluation of the functions was carried out by Lommel himself in 1886. A more comprehensive table was published by Dekanosidze [164] in 1956, again for only real values of $w$ and $s$. Additional studies for real values have been carried out by Rayleigh [165], Hopkins [166], Conrady [167], Buxton [168], Boersma [169] and Rybner [170]. The functions of purely imaginary argument, denoted by

$$Y_n(w,s) \equiv i^{-n} U_n(iw, is) = \sum_{m=0}^{\infty} \big(\frac{w}{s}\big)^{2m+n} I_{2m+n}(s) \ , \tag{D.5}$$

$$\Theta_n(w,s) \equiv i^{-n} V_n(iw, is) = \sum_{m=0}^{\infty} \big(\frac{w}{s}\big)^{-(2m+n)} I_{-(2m+n)}(s) \ , \tag{D.6}$$

have been tabulated by Kuznetsov [171] and Bark and Kuznetsov [172]. To our knowledge, no one has studied the "modified" functions which we define to be

$$\Lambda_n(w,s) \ \equiv \ i^{-n} U_n(iw, s) = \sum_{m=0}^{\infty} \Big(\frac{w}{s}\Big)^{2m+n} J_{2m+n}(s) \ , \tag{D.7}$$

$$\Xi_n(w,s) \ \equiv \ i^{-n} V_n(iw, s) = \sum_{m=0}^{\infty} \Big(\frac{w}{s}\Big)^{-2m-n} J_{-2m-n}(s) \ . \tag{D.8}$$

The numerical evaluation of the modified functions is presented in Appendix E along with a new asymptotic expansion. The treatment in Appendix E applies to complex values of $w$ so the formula therein may be used to calculate Lommel functions of two real variables.

Now we turn our attention to deriving a few properties of the Lommel functions for the special case in which the arguments are of the form

$$\begin{aligned} w &= \beta(\tau - |z|) \ , \\ s &= \sqrt{\tau^2 - z^2} \ , \end{aligned} \tag{D.9}$$

with $\beta$ a complex constant independent of $\tau$ and $z$. We restrict ourselves to the $U_n(w,s)$ Lommel functions although similar relations exist for the $V_n(w,s)$ functions and may be found in the literature [91, 162, 163]. Using the recurrence relation for Bessel functions [173], and the defining series of Lommel functions,

$$U_n(w,s) = \sum_{m=0}^{\infty} (-1)^{2m+n} J_{2m+n}(s) \ , \tag{D.10}$$

one may derive the following:

$$U_n(w,s) = \left(\frac{w}{s}\right)^n J_n(s) - U_{n+2}(w,s) \ , \tag{D.11}$$

$$\frac{\partial U_n(w,s)}{\partial s} = -\frac{s}{w} U_{n+1}(w,s) \ , \tag{D.12}$$

$$\frac{\partial U_n(w,s)}{\partial w} = \frac{1}{2} U_{n-1}(w,s) + \frac{1}{2}\left(\frac{s}{w}\right)^2 U_{n+1}(w,s) \ . \tag{D.13}$$

For the variables $(w,s)$ as defined in Eq. (D.9) we have

$$\frac{\partial U_n(\beta w,s)}{\partial |z|} = -\frac{1}{2}\Big[\beta U_{n-1}(\beta w,s) + \frac{1}{\beta} U_{n+1}(\beta w,s)\Big] \ , \tag{D.14}$$

$$\frac{\partial U_n(\beta w,s)}{\partial \tau} = -\frac{1}{2}\Big[\beta U_{n-1}(\beta w,s) - \frac{1}{\beta} U_{n+1}(\beta w,s)\Big] \ , \tag{D.15}$$

$$\frac{\partial^2 U_n(\beta w,s)}{\partial^2 |z|^2} = \frac{1}{4}\Big[\beta^2 U_{n-2}(\beta w,s) + 2U_n(\beta w,s) + \frac{1}{\beta^2} U_{n+2}(\beta w,s)\Big] \ , \tag{D.16}$$

$$\frac{\partial^2 U_n(\beta w,s)}{\partial^2 \tau^2} = \frac{1}{4}\Big[\beta^2 U_{n-2}(\beta w,s) - 2U_n(\beta w,s) + \frac{1}{\beta^2} U_{n+2}(\beta w,s)\Big] \ , \tag{D.17}$$

Subtracting Eq. (D.16) from Eq. (D.17) we have

$$\frac{\partial^2 U_n(\beta w,s)}{\partial^2 \tau^2} - \frac{\partial^2 U_n(\beta w,s)}{\partial^2 |z|^2} = -U_n(\beta w,s) \ . \tag{D.18}$$

Therefore $U_n(\beta w,s)$ is a solution of the "massive" Klein-Gordon equation (at least in the positive half-space since $|z| > 0$).

The above properties hold for arbitrary complex $w$ and $s$. We now focus on the modified functions in which $w$ is pure imaginary. Introducing the notation

$$\Lambda_n(w,s) = i^{-n} U_n(iw,s) \ , \tag{D.19}$$

with $w$ and $s$ given by Eq. (D.9), we consider the limit in which $z \to 0$ for which

$$\frac{w}{s} = \sqrt{\frac{\tau - |z|}{\tau + |z|}} \to 1 \ . \tag{D.20}$$

For $n$ even we have

$$\Lambda_{2n}(\tau,\tau) = -\sum_{m=1}^{n-1} J_{2m}(\tau) + \frac{1 - J_0(\tau)}{2} \ . \tag{D.21}$$

For odd $n$ we use an integral representation

$$\Lambda_{2n+1}(\tau,\tau) = -\sum_{m=0}^{n-1} J_{2m+1}(\tau) + \frac{1}{2}\int_0^\infty dx J_0(x) \ , \tag{D.22}$$

or in terms of Struve functions [174],

$$\Lambda_{2n+1}(\tau,\tau) = -\sum_{m=0}^{n-1} J_{2m+1}(\tau) + \frac{1}{2}\Big\{\tau J_0(\tau) + \frac{\pi\tau}{2}[J_1(\tau)\mathbf{H}_0(\tau) - J_0(\tau)\mathbf{H}_1(\tau)]\Big\} \ . \tag{D.23}$$

Finally we consider the limiting case of $\tau = |z|$ , i.e. $s = w = 0$. Since for all $n \geq 1$ $J_n(0) = 0$ , we have

$$\begin{aligned} \Lambda_0(0,0) &= 1 \ , \\ \Lambda_n(0,0) &= 0 \quad n \geq 1 \ . \end{aligned} \tag{D.24}$$

While some of the properties (especially D.18) derived above are useful for the actual derivation of the Green functions in Chapter 4, they are most useful when checking the analytic expressions by operating on them with the differential operator

$$\partial_{tt} - \partial_{xx} + U''[\phi_c(x)] \ . \tag{D.25}$$

# Appendix E

# Numerical Evaluation and Asymptotic Forms of Modified Lommel Functions of Two Variables

Numerical evaluation of the Green functions derived in Chapter 4 requires an evaluation of the modified Lommel functions. Although Lommel functions of two real variables [164] and two purely imaginary variables [172] have been studied, to our knowledge no one has yet considered the modified functions. Below we present methods which are valid for $w$ complex and $s$ real (since we start by considering the modified functions and w may be complex, our methods also include the case of two real variables). Representing the first argument as $\beta w$, where $|\beta| = 1$ and $w$ and $s$ are real, we have for the defining series

$$\Lambda_n(\beta w, s) = \sum_{m=0}^{\infty} \left( \frac{\beta w}{s} \right)^{2m+n} J_{2m+n}(s) \ , \tag{E.1}$$

from which we deduce the symmetries

$$\begin{aligned} \Lambda_n(-\beta w, s) &= (-1)^n \Lambda_n(\beta w, s) \ , \\ \Lambda_n(\beta w, -s) &= \Lambda_n(\beta w, s) \ . \end{aligned} \tag{E.2}$$

From Eqs. (E.2) we see that we need only investigate the first quadrant of the $w$–$s$ plane. Another relationship exists which allows us to further restrict our attention to the angular region $(0, \pi/4)$, i.e. the first octant. We obtain this property by recalling the generating function for Bessel functions [175]:

$$e^{\frac{s}{2}[\beta\kappa - \frac{1}{\beta\kappa}]} = \sum_{m=-\infty}^{\infty} (\beta\kappa)^m J_m(s) \ , \tag{E.3}$$

where $\kappa \equiv w/s$. Using the symmetry of the Bessel functions about the origin we have,

$$\begin{aligned}
\sinh\Big[\frac{s}{2}\Big(\beta\kappa - \frac{1}{\beta\kappa}\Big)\Big] &= \sum_{m=-\infty}^{\infty} (\beta\kappa)^{2m} J_{2m}(s) \ , \\
\cosh\Big[\frac{s}{2}\Big(\beta\kappa - \frac{1}{\beta\kappa}\Big)\Big] &= \sum_{m=-\infty}^{\infty} (\beta\kappa)^{2m+1} J_{2m+1}(s) \ .
\end{aligned} \tag{E.4}$$

Next we note that

$$\Lambda_n\Big(\frac{s^2}{\beta w}, s\Big) = \sum_{m=0}^{\infty} (\frac{s}{\beta w})^{2m+n} J_{2m+n}(s) \ , \tag{E.5}$$

which leads us to

$$\begin{aligned}
\sinh\Big[\frac{s}{2}\Big(\beta\kappa - \frac{1}{\beta\kappa}\Big)\Big] &= \Lambda_1(\beta w, s) - \Lambda_1\Big(\frac{s^2}{\beta w}, s\Big) \ , \\
\cosh\Big[\frac{s}{2}\Big(\beta\kappa - \frac{1}{\beta\kappa}\Big)\Big] &= J_0(s) + \Lambda_0(\beta w, s) + \Lambda_0\Big(\frac{s^2}{\beta w}, s\Big) \ .
\end{aligned} \tag{E.6}$$

From Eqs. (E.6) we see that we have a relationship which allows us to consider only the region of the first quadrant of the $s$–$w$ plane in which $w/s < 1$, namely the first octant. In this region the series definition (E.1) converges uniformly, however that rate of convergence is very slow when one approaches $w/s = 1$. By comparison with the geometric series we see that since $J_n(s) < 1 \ \forall \ n$ , we have as an error estimate for truncation after $N$ terms

$$R_N < \frac{\kappa^{2N}}{1-\kappa^2} \ , \tag{E.7}$$

We note that the error estimate in Eq. (E.7) is very crude as it does not take into account the decaying nature of the Bessel functions, however it suffices for our calculations.

As $w/s \to 1$, the number of terms in the series needed to attain a given accuracy becomes unreasonably large. For values of $\kappa = w/s$ larger than some $\kappa_0$, we turn to an asymptotic expansion [176] of the modified Lommel functions. We begin by following Mayall's [177] procedure for obtaining an integral representation for the Lommel functions by substitution of an integral representation for the Bessel functions into the series and summing the series explicitly. We restrict ourselves to deriving expressions for $\Lambda_0$ and $\Lambda_1$. For small $n$ the asymptotic expansion for $\Lambda_n$ may be obtained from the recurrence relation for Lommel functions. The large $n$ limit has not yet been examined.

Starting with the integral representation for Bessel functions

$$J_{2m}\left(s\right) = \frac{(-1)^m}{\pi}\int_0^\pi d\theta e^{is\cos(\theta)}\cos\left(2m\theta\right) \ , \tag{E.8}$$

we have

$$\begin{aligned}
\Lambda_0(\beta w, s) &= \sum_{m=0}^{\infty}(\beta\kappa)^{2m}(-1)^m\frac{1}{\pi}\int_0^\pi d\theta e^{is\cos(\theta)}\cos(2m\theta) \ , &\text{(E.9)}\\
&= \frac{1}{\pi}\int_0^\pi d\theta\frac{1+(\beta\kappa)^2\cos(2\theta)}{1+2(\beta\kappa)^2\cos(2\theta)+(\beta\kappa)^4}e^{is\cos(\theta)} \ , &\text{(E.10)}\\
&= \frac{J_0(s)}{2}+\frac{1-(\beta\kappa)^4}{2\pi}\int_0^\pi d\theta\frac{e^{is\cos(\theta)}}{1+2(\beta\kappa)^2\cos(2\theta)+(\beta\kappa)^4} \ , &\text{(E.11)}\\
&= \frac{J_0(s)}{2}+\sigma_1(\beta,\kappa)\frac{\epsilon(\beta,\kappa)}{\pi}\int_0^\pi d\theta\frac{e^{is\cos(\theta)}}{\epsilon^2(\beta,\kappa)+\cos^2(\theta)} \ , &\text{(E.12)}
\end{aligned}$$

where

$$\begin{aligned}
\epsilon(\beta,\kappa) &\equiv \frac{1-(\beta\kappa)^2}{2\beta\kappa} \ , \\
\sigma_1(\beta,\kappa) &\equiv \frac{1+(\beta\kappa)^2}{4\beta\kappa} \ , &\text{(E.13)}
\end{aligned}$$

and uniform convergence of the sum has been used. Similarly we may write

$$\Lambda_1(\beta w, s) = -\sigma_2(\beta,\kappa)\frac{d}{ds}\frac{\epsilon(\beta,\kappa)}{\pi}\int_0^\pi d\theta\frac{e^{is\cos(\theta)}}{\epsilon^2(\beta,\kappa)+\cos^2(\theta)} \ , \tag{E.14}$$

with

$$\sigma_2(\beta,\kappa) \equiv \frac{1+(\beta\kappa)^2}{4}+\frac{\beta\kappa[1+\epsilon^2(\beta,\kappa)]}{2\epsilon(\beta,\kappa)} \ . \tag{E.15}$$

At this point, Mayall's method no longer applies (unless $\beta = \pm i$) and we turn to an alternate derivation.

The integral $I(\epsilon, s)$ given by

$$I(\epsilon, s) = \frac{\epsilon}{\pi}\int_0^\pi d\theta\frac{e^{is\cos(\theta)}}{\epsilon^2+\cos^2(\theta)} \ , \tag{E.16}$$

which occurs in Eqs. (E.12) and (E.14), is strong function of $\epsilon$ since in the limit as $\epsilon \to 0$ ($w/s \to 1$), we obtain a delta function. Other major contributions occur

Figure E.1: Contour for the asymptotic values of the Lommel functions.

at the stationary points $\theta = 0, \pi$. To evaluate $I(\epsilon, s)$, we substitute $t = \cos(\theta)$, deform the contour and represent the integrals as a residue which captures the strong $\epsilon$ behavior, plus two integrals for which asymptotic expansions are easily derived. Substituting we have

$$
\begin{aligned}
I(\epsilon, s) &= \frac{\epsilon}{\pi} \int_{-1}^{1} d\theta \frac{e^{ist}}{(\epsilon^2 + t^2)\sqrt{1 - t^2}} \ , && \text{(E.17)} \\
&= \frac{\epsilon}{\pi} \Bigg\{ 2\pi i Res[f(z), i\epsilon] - \int_{c_1} dz \frac{e^{isz}}{(\epsilon^2 + z^2)\sqrt{1 - z^2}} \\
&\qquad - \int_{c_3} dz \frac{e^{isz}}{(\epsilon^2 + z^2)\sqrt{1 - z^2}} \Bigg\} \ , && \text{(E.18)}
\end{aligned}
$$

where $f(z)$ is given by

$$
f(z) = \frac{\epsilon}{\pi} \frac{e^{isz}}{(\epsilon^2 + z^2)\sqrt{1 - z^2}} \ , \qquad \text{(E.19)}
$$

and the contours are shown in Figure E.1. We have used the fact that as $\delta \to 0$ and $y_0 \to \infty$, the contributions from the contours $c_{\delta 1}$, $c_{\delta 2}$ and $c_2$ vanish by Jordan's lemma. Evaluating the residue and shifting the variables, we have

$$\begin{aligned} I(\epsilon, s) &= \frac{e^{-\epsilon s}}{\sqrt{1+\epsilon^2}} - \int_0^{i\infty} dz \frac{e^{isz} e^{is}}{[\epsilon^2 + (z+1)^2]\sqrt{1-(z+1)^2}} \\ &- \int_{i\infty}^{0} dz \frac{e^{isz} e^{-is}}{[\epsilon^2 + (z-1)^2)]\sqrt{1-(z-1)^2}} \ , \qquad \text{(E.20)} \\ &= \frac{e^{-\epsilon s}}{\sqrt{1+\epsilon^2}} - \frac{\epsilon}{\pi}[J + J^*] \ , \qquad \text{(E.21)} \end{aligned}$$

where

$$\begin{aligned} J &\equiv ie^{is} \int_0^\infty dy \frac{e^{-sy}}{[\epsilon^2 + (iy+1)^2]\sqrt{1-(iy+1)^2}} \ , \qquad \text{(E.22)} \\ &= 2ie^{is} \int_0^\infty dx \frac{e^{-sx^2}}{[\epsilon^2 + (ix^2+1)^2)\sqrt{x^2-2i}} \ , \qquad \text{(E.23)} \end{aligned}$$

As written in Eq. (E.23), $J$ is in one of Dingle's [178] standard integral forms which has as an asymptotic expansion

$$J \approx 2ie^{is}\sqrt{\frac{\pi}{2F_{01}}} e^{-F_0} \sum_{n=0}^{\infty} Q_r \ , \qquad \text{(E.24)}$$

where

$$\begin{aligned} Q_0 &= G_0 \ , \\ Q_1 &= \frac{-\sqrt{2}}{3\sqrt{\pi} F_2^{\frac{3}{2}}}[-3G_1F_2] \ , \\ Q_2 &= \frac{1}{24F_2^3}[12G_2F_2^2] \ , \\ Q_3 &= \frac{-\sqrt{2}}{135\sqrt{\pi} F_2^{\frac{9}{2}}}[-45G_4F_2^4] \ , \\ Q_4 &= \frac{1}{1152F_2^6}[144G_4F_2^4] \ , \qquad \text{(E.25)} \\ F_\nu &= \left(\frac{d}{dx}\right)^\nu sx^2 \ , \qquad \text{(E.26)} \\ G_\nu &= \left(\frac{d}{dx}\right)^\nu \frac{1}{[\epsilon^2 + (ix^2+1)^2]\sqrt{x^2-2i}} \ . \qquad \text{(E.27)} \end{aligned}$$

Carrying out the derivatives, we have, including up to $Q_4$

$$J + J^* = -\frac{2}{1+\epsilon^2}\sqrt{\frac{2}{\pi s}}\Big\{ \quad \cos(s-\frac{\pi}{4})\big[\frac{1}{2} + \frac{R_4(\beta,\kappa)}{(8s)^2}\big] + \sin(s-\frac{\pi}{4})\big[\frac{R_2(\beta,\kappa)}{(8s)}\big]\Big\} + O(s^{-\frac{7}{2}}) \ , \quad \text{(E.28)}$$

where

$$R_2(\beta,\kappa) = \frac{9+\epsilon^2(\beta,\kappa)}{2[1+\epsilon^2(\beta,\kappa)]} \ , \quad \text{(E.29)}$$

$$R_4(\beta,\kappa) = -\frac{9}{4} + \frac{12}{1+\epsilon^2(\beta,\kappa)} - \frac{96}{(1+\epsilon^2(\beta,\kappa))^2} \ . \quad \text{(E.30)}$$

With Eq. (E.28) we now have an asymptotic expansion for $I(\epsilon, s)$, which leads to the following expressions for $\Lambda_0(\beta w, s)$ and $\Lambda_1(\beta w, s)$ :

$$\begin{aligned} \Lambda_0(\beta w, s) &\approx \frac{J_0(s)}{2} + \sigma_1(\beta,\kappa)\frac{e^{-\epsilon(\beta,\kappa)s}}{\sqrt{1+\epsilon^2(\beta,\kappa)}} \\ &+ \sigma_1(\beta,\kappa)\sqrt{\frac{2}{\pi s}}\frac{\epsilon(\beta,\kappa)}{1+\epsilon^2(\beta,\kappa)}\Big\{\cos(s-\frac{\pi}{4})\big[1+\frac{2R_4(\beta,\kappa)}{(8s)^2}\big] \\ &+ \sin(s-\frac{\pi}{4})\big[\frac{2R_2(1,\kappa)}{8s}\big]\Big\} + \frac{\sigma_1(\beta,\kappa)}{\sqrt{1+\epsilon^2(\beta,\kappa)}}O(s^{-\frac{7}{2}}), \end{aligned} \quad \text{(E.31)}$$

$$\begin{aligned} \Lambda_1(\beta w, s) &\approx \frac{\epsilon(\beta,\kappa)\sigma_2(\beta,\kappa)}{\sqrt{1+\epsilon^2(\beta,\kappa)}}\Big\{e^{-\epsilon(\beta,\kappa)s} - \frac{1}{\sqrt{1+\epsilon^2(\beta,\kappa)}}\sqrt{\frac{2}{\pi s}} \times \\ &\times \Big[\cos(s-\frac{\pi}{4})\Big(\frac{2[R_2(\beta,\kappa)-2]}{8s} - 40\frac{R_4(\beta,\kappa)}{(8s)^3}\Big) \\ &- \sin(s-\frac{\pi}{4})\Big(1+\frac{2[R_4(\beta,\kappa)+12R_2(\beta,\kappa)]}{(8s)^2}\Big)\Big]\Big\} \\ &+ \frac{\epsilon(\beta,\kappa)\sigma_2(\beta,\kappa)}{1+\epsilon^2(\beta,\kappa)}O(s^{-\frac{9}{2}}) \ , \end{aligned} \quad \text{(E.32)}$$

# Appendix F

# Thermal Averages and Correlation Functions

In Chapter 6 we require the thermal average of several functions of the normal mode amplitudes $b_k$. In general the $b_k$'s are complex and we use for convenience the following definition for the averages

$$\langle F(b_q, b^*_{q'})\rangle = \frac{\prod\limits_k \int\limits_{-\infty}^{\infty} db_k \int\limits_{-\infty}^{\infty} db^*_k \; F(b_q, b^*_{q'}) e^{-\beta\omega_k |b_k|^2}}{\prod\limits_k \int\limits_{-\infty}^{\infty} db_k \int\limits_{-\infty}^{\infty} db^*_k \; e^{-\beta\omega_k |b_k|^2}} \tag{F.1}$$

From this definition it is clear that $\langle b_k\rangle = \langle b^*_k\rangle = 0$. However, to see that quantities such as $\langle b^2_k\rangle = \langle {b_k}^{*2}\rangle = 0$ it is useful to write the average in terms of the real and imaginary parts, for example

$$\langle b^2_k\rangle = \langle {b^R_k}^2 + 2b^R_k b^I_k - {b^I_k}^2\rangle \; . \tag{F.2}$$

The average in terms of the real and imaginary parts becomes

$$\langle F(b_q, b^*_{q'})\rangle = \frac{\prod\limits_k \int\limits_{-\infty}^{\infty} db^R_k \int\limits_{-\infty}^{\infty} db^I_k \; F(b_q, b_{q'}) e^{-\beta\omega_k ({b^R_k}^2 + {b^I_k}^2)}}{\prod\limits_k \int\limits_{-\infty}^{\infty} db^R_k \int\limits_{-\infty}^{\infty} db^I_k \; e^{-\beta\omega_k ({b^R_k}^2 + {b^I_k}^2)}} \; , \tag{F.3}$$

from which we can see that the cross term in Eq. (F.2) is zero and the quadratic terms are equal.

In general the complex notation is easier to handle which may be illustrated by the ease with which $\langle b_q b^*_q\rangle$ is computed:

$$\langle b_q b^*_q\rangle \;=\; \frac{\prod\limits_k \int\limits_{-\infty}^{\infty} db_k \int\limits_{-\infty}^{\infty} db^*_k |b_q|^2 e^{-\beta\omega_k |b_k|^2}}{\prod\limits_k \int\limits_{-\infty}^{\infty} db_k \int\limits_{-\infty}^{\infty} db^*_k e^{-\beta\omega_k |b_k|^2}} \; , \tag{F.4}$$

$$= \frac{\int_0^\infty |b_q|^2 e^{-\beta\omega_q |b_q|^2} d|b_q|^2}{\int_0^\infty e^{-\beta\omega_q |b_q|^2} d|b_q|^2} \;, \tag{F.5}$$

$$= \frac{T}{\omega_q} \;, \tag{F.6}$$

where we have made a transformation to polar coordinates and taken $k_B$ to be 1. With these averages computed we can compute some of the more complicated averages and correlations. The first of these is the average of $\psi^2$ which is

$$\begin{aligned}
\langle \psi^2(x,t) \rangle &= \Bigg\langle \sum_{k_1,k_2} \Big[ b_{k_1} f_{k_1}(x) e^{-i\omega_{k_1} t} + b^*_{k_1} f^*_{k_1}(x) e^{i\omega_{k_1} t} \Big] \times \\
&\times \Big[ b_{k_2} f_{k_2}(x) e^{-i\omega_{k_2} t} + b^*_{k_2} f^*_{k_2}(x) e^{i\omega_{k_2} t} \Big] \Bigg\rangle \;, && \text{(F.7)} \\
&= \sum_k \frac{\langle b_k b^*_k \rangle |f_k(x)|^2}{\omega_k} \;, && \text{(F.8)} \\
&= T \sum_k \frac{|f_k(x)|^2}{\omega_k^2} \;. && \text{(F.9)}
\end{aligned}$$

The sum in Eq. (F.9) is exactly the static Green function. Using the sine-Gordon static Green function [115] we can write

$$\langle \psi^2(x,t) \rangle = \frac{T}{2} \Big( 1 - \frac{1}{2} \mathrm{sech}^2 x \Big) \;. \tag{F.10}$$

Next we use the fact that the functions $f_k(x)$ have the following symmetry

$$f_k(-x) = \pm f_{-k}(x) \;, \tag{F.11}$$

which tells us that

$$\langle \psi^2(-x,t) \rangle = \langle \psi^2(x,t) \rangle \;. \tag{F.12}$$

Next using the fact that $U'''[\phi_c(-x)] = -U'''[\phi_c(x)]$ and $\phi'_c(-x) = \phi'_c(x)$ we have [recall Eq. (6.1.3) for $F_\psi$]

$$\langle F_\psi \rangle = \frac{1}{2M_0} \int_{-\infty}^{\infty} U'''[\phi_c(x)] \phi'_c(x) \langle \psi^2(x,t) \rangle = 0 \;. \tag{F.13}$$

Also since $\eta_\psi$ [see Eq. (6.1.2)] is linear in $\psi$ we have

$$\langle \eta_\psi \rangle = 0 \;. \tag{F.14}$$

Now we go on to compute the correlation functions $\langle \eta_\psi(t)\ \eta_\psi(t')\rangle$ and $\langle F_\psi(t)\ F_\psi(t')\rangle$ . First we consider the $\eta$ correlation function which we write as

$$\begin{aligned}
\langle \eta_\psi(t)\ \eta_\psi(t')\rangle &= \frac{4}{M_0^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \phi_c''(x)\phi_c''(x') \times \\
&\quad \times \sum_k \left[\frac{\omega_k}{2}\langle b_k b_k^*\rangle f_k(x) f_k^*(x') e^{-i\omega_k(t-t')} + H.C.\right] \qquad (F.15)\\
&= \frac{2T}{M_0^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \phi_c''(x)\phi_c''(x') \times \\
&\quad \times \sum_k \left[f_k(x) f_k^*(x') e^{-i\omega_k(t-t')} + H.C.\right] \qquad (F.16)\\
&= \frac{4T}{M_0^2}\sum_k \left|\int dx \phi_c''(x) f_k(x)\right|^2 \cos[\omega_k(t-t')]\ . \qquad (F.17)
\end{aligned}$$

For $t - t' > 1/\omega_{k_0}$ this correlation function decays rapidly where $\omega_{k_0}$ is the lowest frequency and $H.C.$ means Hermitian conjugate.

The last correlation function computed is $\langle F_\psi(t)\ F_\psi(t')\rangle$ where $F_\psi(t)$ in terms of normal mode amplitudes is given by

$$\begin{aligned}
F_\psi &= \frac{1}{2M_0}\int dx U'''[\phi_c(x)]\phi_c'(x) \sum_{k_1,k_2}\frac{1}{\sqrt{4\omega_{k_1}\omega_{k_2}}}\Big\{ b_{k_1} b_{k_2} f_{k_1}(x) f_{k_2}(x) e^{-i(\omega_{k_1}+\omega_{k_2})t} \\
&+ \qquad b_{k_1} b_{k_2}^* f_{k_1}(x) f_{k_2}^*(x) e^{-i(\omega_{k_1}-\omega_{k_2})t} + \ H.C.\Big\}\ . \qquad (F.18)
\end{aligned}$$

In doing the average only those terms which have two $b_k$'s and $b_k^*$'s are nonzero; therefore we write

$$\langle F_\psi(t)\ F_\psi(t')\rangle = \frac{1}{2M_0}\int dx U'''[\phi_c(x)]\phi_c'(x)\int dx' U'''[\phi_c(x')]\phi_c'(x')\Big[A + B + H.C.\Big]\ , \qquad (F.19)$$

with $A$ and $B$ given by

$$A = \sum_{k_1,k_2,k_3,k_4}\frac{\langle b_{k_1} b_{k_2} b_{k_3}^* b_{k_4}^*\rangle}{\sqrt{\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4}}} f_{k_1}(x) f_{k_2}(x) f_{k_3}^*(x') f_{k_4}^*(x') e^{-i(\omega_{k_1}+\omega_{k_2})t + i(\omega_{k_3}+\omega_{k_4})t'}\ , \qquad (F.20)$$

$$B = \sum_{k_1,k_2,k_3,k_4}\frac{\langle b_{k_1} b_{k_2}^* b_{k_3} b_{k_4}^*\rangle}{\sqrt{\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4}}} f_{k_1}(x) f_{k_2}^*(x) f_{k_3}(x') f_{k_4}^*(x') e^{-i(\omega_{k_1}-\omega_{k_2})t - i(\omega_{k_3}-\omega_{k_4})t'}\ . \qquad (F.21)$$

First we consider the $A$ term. The averages of the four $b$ factors is zero unless $k_1 = k_3, k_2 = k_4$ or $k_1 = k_4, k_2 = k_3$, both of which are the same due to relabeling and therefore we restrict ourselves to the former. First we consider $k_1 = k_3, k_2 = k_4$ but $k_1 \neq k_4$ in which the average over the $b_{k_i}$ factors yields

$$\langle |b_{k_1}|^2 |b_{k_3}|^2 \rangle = \frac{T}{\omega_{k_1}} \frac{T}{\omega_{k_2}} \ . \tag{F.22}$$

Therefore this contribution to $A$ can be written as

$$2 \sum_{k_1,k_2} \frac{T^2}{\omega_{k_1}^2 \omega_{k_2}^2} f_{k_1}(x) f_{k_1}^*(x') f_{k_2}(x) f_{k_2}^*(x') e^{-i(\omega_{k_1}+\omega_{k_2})(t-t')} \ , \tag{F.23}$$

where the factor of 2 is due to the $k_1 = k_4, k_2 = k_3$ term. In the case that $k_1 = k_2 = k_3 = k_4$ we must evaluate

$$\langle |b_k|^4 \rangle = \frac{\int_0^\infty |b_k|^4 e^{-\beta\omega_k |b_k|^2} d|b_k|^2}{\int_0^\infty e^{-\beta\omega_k |b_k|^2} d|b_k|^2} \ , \tag{F.24}$$

$$= \frac{2T^2}{\omega_k^2} \ , \tag{F.25}$$

which leads to the following contribution to $A$

$$2 \sum_{k_1,k_2} \delta_{k_1,k_2} \frac{T^2}{\omega_{k_1}^2 \omega_{k_2}^2} f_{k_1}(x) f_{k_1}^*(x') f_{k_2}(x) f_{k_2}^*(x') e^{-i(\omega_{k_1}+\omega_{k_2})(t-t')} \ , \tag{F.26}$$

which allows us to write for $A$

$$A = 2 \sum_{k_1,k_2} \frac{T^2}{\omega_{k_1}^2 \omega_{k_2}^2} f_{k_1}(x) f_{k_1}^*(x') f_{k_2}(x) f_{k_2}^*(x') e^{-i(\omega_{k_1}+\omega_{k_2})(t-t')} \ , \tag{F.27}$$

Similarly for the $B$ term we consider the terms with $k_1 = k_2, k_3 = k_4$ but $k_1 \neq k_3$ (not the same as $k_1 = k_4, k_2 = k_3$) which yields the contribution

$$\sum_{k_1 \neq k_3} \frac{T^2}{\omega_{k_1}\omega_{k_3}} |f_{k_1}(x)|^2 |f_{k_3}(x')|^2 = \sum_{k_1,k_3} \frac{T^2}{\omega_{k_1}\omega_{k_3}} |f_{k_1}(x)|^2 |f_{k_3}(x')|^2 - \sum_k \frac{T^2}{\omega_k^2} |f_k(x)|^2 |f_k(x')|^2 \ . \tag{F.28}$$

Before we compute the other contributions to $B$ we note that when the required integrals over space are done to complete the calculation of $\langle F_\psi(t)\ F_\psi(t') \rangle$, the first term of Eq. (F.27) may be written as

$$\left[ \int_{-\infty}^{\infty} dx U'''[\phi_c(x)] \phi_c'(x) \sum_k \frac{T}{\omega_k^2} |f_k(x)|^2 \right]^2 \ , \tag{F.29}$$

which is zero since the integrand is odd upon the interchange $x \to -x$ , $k \to -k$. The term for which $k_1 = k_4, k_2 = k_3$ but $k_1 \neq k_2$ can be written as

$$\sum_{k_1 \neq k_2} \frac{T^2}{\omega_{k_1}\omega_{k_2}} f_{k_1}(x) f^*_{k_1}(x') f^*_{k_2}(x) f_{k_2}(x') e^{-i(\omega_{k_1}-\omega_{k_2})(t-t')} \ . \tag{F.30}$$

Finally, evaluation of the $k_1 = k_2 = k_3 = k_4$ term yields exactly $-1/2$ of the second term in Eq. (F.27), which when combined with Eq. (F.29) yields

$$B = \sum_{k_1,k_2} \frac{T^2}{\omega^2_{k_1}\omega^2_{k_2}} f_{k_1}(x) f^*_{k_1}(x') f^*_{k_2}(x) f_{k_2}(x') e^{-i(\omega_{k_1}-\omega_{k_2})(t-t')} \ . \tag{F.31}$$

Next we use the fact that $f_{-k}(x) = \pm f^*_k(x)$ which allows us to write

$$B = \sum_{k_1,k_2} \frac{T^2}{\omega^2_{k_1}\omega^2_{k_2}} f_{k_1}(x) f^*_{k_1}(x') f_{k_2}(x) f^*_{k_2}(x') e^{-i(\omega_{k_1}-\omega_{k_2})(t-t')} \ , \tag{F.32}$$

which is the same as $A/2$ except for the exponential in time. Combining these factors we have

$$\begin{aligned} \langle F_\psi(t)\ F_\psi(t')\rangle &= \frac{1}{4M_0^2} \int dx U'''[\phi_c(x)]\phi'_c(x) \int dx' U'''[\phi_c(x')]\phi'_c(x') \times \\ &\times \sum_{k_1,k_2} \frac{T^2}{\omega^2_{k_1}\omega^2_{k_2}} f_{k_1}(x) f^*_{k_1}(x') f_{k_2}(x) f^*_{k_2}(x') \times \\ &\times \left\{ e^{-i(\omega_{k_1}+\omega_{k_2})(t-t')} + e^{-i(\omega_{k_1}-\omega_{k_2})(t-t')} \right\} + H.C. \end{aligned} \tag{F.33}$$

We can obtain further simplification by recalling that the functions $f_k(x)$ obey

$$-f''_k(x) + U''[\phi_c(x)] f_k(x) = \omega^2_k f_k(x) \tag{F.34}$$

which allows us to rewrite the integrals in Eq. (F.33) as

$$\int dx U'''[\phi_c(x)]\phi'_c(x) f_{k_1}(x) f_{k_2}(x)$$

$$= \int dx f_{k_1}(x) f_{k_2}(x) \frac{dU''[\phi_c(x)]}{dx} \tag{F.35}$$

$$= -\int dx U''[\phi_c(x)] \Big[ f'_{k_1}(x) f_{k_2}(x) + f_{k_1}(x) f'_{k_2}(x) \Big] \tag{F.36}$$

$$= \int \Big\{ f'_{k_1}(x) \Big[ f''_{k_2}(x) + \omega^2_{k_2} f_{k_2} \Big] + f'_{k_2}(x) \Big[ f''_{k_1}(x) + \omega^2_{k_1} f_{k_1} \Big] \Big\} dx \tag{F.37}$$

$$= \int \Big\{ \frac{d}{dx} \Big[ f'_{k_1}(x) f'_{k_2}(x) \Big] + (\omega^2_{k_1} - \omega^2_{k_2}) f_{k_1}(x) f'_{k_2}(x) + \frac{d}{dx} \Big[ f_{k_1}(x) f'_{k_2}(x) \Big] \omega^2_{k_2} \Big\} dx \tag{F.38}$$

$$= (\omega^2_{k_1} - \omega^2_{k_2}) \int dx f_{k_1}(x) f'_{k_2}(x) \ , \tag{F.39}$$

where the surface terms vanish by periodic boundary conditions. Finally we have

$$\begin{aligned} \langle F_\psi(t)\ F_\psi(t')\rangle &= \frac{T^2}{2M_0^2}\sum_{k_1,k_2}\frac{(\omega_{k_1}^2-\omega_{k_2}^2)^2}{\omega_{k_1}^2\omega_{k_2}^2}\Big|\int dx f_{k_1}(x)f'_{k_2}(x)\Big|^2\times \\ &\times \Big[\cos[(\omega_{k_1}+\omega_{k_2})(t-t')]+\cos[(\omega_{k_1}-\omega_{k_2})(t-t')]\Big]\ . \end{aligned} \tag{F.40}$$

# Appendix G

# Functional Derivatives in Terms of Kink Variables

In order to derive the Fokker-Planck equation in Appendix H for the kink variables $X$ and $p$ we need to have expressions for the derivatives with respect to $\Phi(x,t)$ and $\Pi_0(x,t)$ in terms of the new variables $\{X, p, \psi, \pi\}$. The fact that this transformation is nontrivial may be seen recalling that the $\psi$ field is constrained to be in the subspace which is perpendicular to $\phi_c(x)$. Therefore, when we take a functional derivative with respect to $\psi$ it must be understood to include only variations in that subspace. To see how to take such "constrained" derivatives, it is useful to examine what is meant when a "regular" functional derivative is taken. Consider for example the derivative of a field $F[\Phi(x,t)]$ with respect to $\Phi(x',t')$

$$\frac{\delta F[\Phi(x,t)]}{\delta\Phi(x',t')} \equiv \lim_{\epsilon\to 0} \frac{F[\Phi(x,t)+\epsilon\delta(x-x')] - F[\Phi(x,t)]}{\epsilon} \; . \tag{G.1}$$

From this definition, it is clear what is meant by a derivative which is constrained to the subspace perpendicular to $\phi_c(x)$, namely

$$\frac{\delta F[\psi(x,t)]}{\delta\psi(x',t')} = \lim_{\epsilon\to 0} \frac{F[\Phi(x,t)+\epsilon\delta(x-x') - \epsilon\frac{\phi_c'(x)\phi_c'(x')}{M_0}] - F[\Phi(x,t)]}{\epsilon} \; . \tag{G.2}$$

In subtracting the "translation mode" term we allow only variations which are in the $\psi$ subspace. In particular we have the following derivatives,

$$\frac{\delta\psi(x,t)}{\delta\psi(x',t')} = \delta(x-x')\delta(t-t') - \frac{\phi_c'(x)\phi_c'(x')}{M_0}\delta(t-t') \; , \tag{G.3}$$

$$\frac{\delta\pi(x,t)}{\delta\pi(x',t')} = \delta(x-x')\delta(t-t') - \frac{\phi_c'(x)\phi_c'(x')}{M_0}\delta(t-t') \; , \tag{G.4}$$

which follow from Eq. (G.2). In writing these "constrained" derivatives, one should be able to avoid the use of Dirac brackets by using the standard Poisson brackets with the derivatives understood to mean the constrained derivatives. As a check we compute the *Poisson* bracket of $\psi(x,t)$ with $\pi(y,t)$ using the constrained derivatives:

$$\begin{aligned}
&\{\psi(x,t),\pi(y,t)\} \\
&= \int_{-\infty}^{\infty} dz \left[\frac{\delta\psi(x,t)}{\delta\psi(z,t)}\frac{\delta\pi(y,t)}{\delta\pi(z,t)} - \frac{\delta\psi(x,t)}{\delta\pi(z,t)}\frac{\delta\pi(y,t)}{\delta\psi(z,t)}\right] , && \text{(G.5)}\\
&= \int_{-\infty}^{\infty} dz \left(\delta(x-z) - \frac{\phi_c'(x)\phi_c'(z)}{M_0}\right)\left(\delta(y-z) - \frac{\phi_c'(y)\phi_c'(z)}{M_0}\right) , && \text{(G.6)}\\
&= \delta(x-y) - 2\frac{\phi_c'(x)\phi_c'(y)}{M_0} + \frac{\phi_c'(x)\phi_c'(y)}{M_0^2}\int_{-\infty}^{\infty} dz\ \phi_c'(z)\phi_c'(z) \ , && \text{(G.7)}\\
&= \delta(x-y) - \frac{\phi_c'(x)\phi_c'(y)}{M_0} , && \text{(G.8)}
\end{aligned}$$

which is exactly the *Dirac* bracket of $\psi(x,t)$ with $\pi(y,t)$.

With the identities (G.3) and (G.4) in hand we proceed to derive the derivatives with respect to $\Phi(x,t)$ and $\Pi_0(x,t)$. This is accomplished by writing the most general transformation between the variables and requiring the identities

$$\begin{aligned}
&\frac{\delta\Phi(x,t)}{\delta\Phi(x',t')} = \delta(x-x')\delta(t-t') && \frac{\delta\Phi(x,t)}{\delta\Pi_0(x',t')} = 0 && \text{(G.9)}\\
&\frac{\delta\Pi_0(x,t)}{\delta\Phi(x',t')} = 0 && \frac{\delta\Pi_0(x,t)}{\delta\Pi_0(x',t')} = \delta(x-x')\delta(t-t') \ . && \text{(G.10)}
\end{aligned}$$

First we consider the $\Phi$ derivative which may be assumed to have the following form which is linear in derivatives with respect to kink variables:

$$\begin{aligned}
\frac{\delta}{\delta\Phi(x',t')} &= \int dt'' A(x',t',t'')\frac{\delta}{\delta X(t'')} + \int dx''dt'' B(x',t',x'',t'')\frac{\delta}{\delta\psi(\zeta'',t'')} \\
&+ \int dt'' C(x',t',t'')\frac{\delta}{\delta p(t'')} + \int dx''dt'' D(x',t',x'',t'')\frac{\delta}{\delta\pi(\zeta'',t'')} \qquad \text{(G.11)}
\end{aligned}$$

with $\zeta$ defined by

$$\zeta \equiv x - X \ . \qquad \text{(G.12)}$$

Operating on $\Phi(x,t)$ with Eq. (G.9) yields

$$\begin{aligned}
\delta(x-x')\delta(t-t') = &-[\phi_c'(\zeta) + \psi'(\zeta,t)]A(x',t',t) + B(x',t',x,t) \\
&- \frac{\phi_c'(\zeta)}{M_0}\int dx''\ B(x',t',x'',t)\phi_c'(\zeta'') \ .
\end{aligned} \qquad \text{(G.13)}$$

Multiplying Eq. (G.13) by $\phi_c'(\zeta)$ and integrating over $\zeta$ gives us

$$A(x',t',t) = -\frac{\phi_c'(\zeta')}{M_0+\xi(t')}\delta(t-t') \ . \tag{G.14}$$

Multiplying Eq. (G.13) by $\psi(\zeta,t)$ and integrating over $\zeta$ gives us

$$\psi(\zeta',t)\delta(t-t') = \int dx'' B(x',t',x'',t)\Big[\psi(\zeta'',t) - \frac{\xi(t)}{M_0}\phi_c'(\zeta'')\Big] \tag{G.15}$$

The solution to this integral equation is

$$\begin{aligned} B(x',t',x'',t'') = \delta(t'-t'')\Bigg\{\delta(x'-x'') \quad &- \quad \frac{\phi_c'(\zeta')\phi_c'(\zeta'')}{M_0} - \frac{\phi_c'(\zeta')\psi'(\zeta'')}{M_0+\xi(t)} \\ &+ \quad \frac{\phi_c'(\zeta')\phi_c'(\zeta'')\xi(t)}{M_0(M_0+\xi(t))}\Bigg\} \ . \end{aligned} \tag{G.16}$$

When this expression for $B$ is substituted into Eq. (G.13) we see that the terms proportional to $\phi_c'(\zeta'')$ are not necessary since the derivative

$$\frac{\delta\psi(\zeta,t)}{\delta\psi(\zeta',t')} \ , \tag{G.17}$$

is manifestly orthogonal to $\phi_c'(\zeta)$. Therefore the expression for $B$ is effectively

$$B(x',t',x'',t'') = \delta(t'-t'')\Bigg\{\delta(x'-x'') - \frac{\phi_c'(\zeta')\psi'(\zeta'')}{M_0+\xi(t)}\Bigg\} \ . \tag{G.18}$$

The functions $C(x',t',t'')$ and $D(x',t',x'',t'')$ are obtained by operating on $\Pi_0(x',t')$ with Eq. (G.11) which yields after quite a bit of algebra

$$\begin{aligned} 0 \quad &= \quad \frac{1}{M_0+\xi(t)}\Bigg[\phi_c'(\zeta')\Pi_0'(x,t) + \phi_c'(\zeta)\Pi_0'(x',t)\Bigg] - \frac{\phi_c'(\zeta')\phi_c'(\zeta)}{(M_0+\xi(t))^2}\int dx \ \Pi_0'\Phi' \\ &- \quad \frac{\phi_c'(\zeta)C(x',t',t)}{M_0+\xi(t)} + D(x',t',x,t) - \frac{\phi_c'(\zeta)}{M_0}\int dx'' D(x',t',x'',t)\phi_c'(\zeta'') \\ &- \quad \frac{\phi_c'(\zeta)}{M_0+\xi(t)}\Bigg[\psi'(\zeta',t) - \frac{\xi(t)}{M_0}\phi_c'(\zeta'')\Bigg] \ . \end{aligned} \tag{G.19}$$

Multiplying Eq. (G.19) by $\phi_c'(\zeta)$ and $\psi(\zeta,t)$ and integrating over $\zeta$ yields

$$\begin{aligned} 0 \quad &= \quad \frac{M_0}{M_0+\xi(t)}\Pi_0'(x',t) + \frac{\phi_c'(\zeta')}{M_0+\xi(t)}\int dx' \ \Pi_0'(x',t)\phi_c'(\zeta') \\ &- \quad \frac{M_0\phi_c'(\zeta')}{(M_0+\xi(t))^2}\int dx' \ \Pi_0'(x',t)\Phi'(x',t) - \frac{M_0}{M_0+\xi(t)}C(x',t',t) \\ &- \quad \frac{M_0}{M_0+\xi(t)}\int dx'' D(x',t',x'',t)\Bigg[\psi'(\zeta'',t) - \frac{\xi(t)}{M_0}\phi_c'(\zeta'')\Bigg] \ , \end{aligned} \tag{G.20}$$

and

$$\begin{aligned}
0 &= -\frac{\xi(t)}{M_0+\xi(t)}C(x',t',t) + \frac{M_0}{M_0+\xi(t)}\int dx'' D(x',t',x'',t)\psi'(\zeta'',t) \\
&- \frac{\xi(t)}{M_0+\xi(t)}\int dx'' D(x',t',x'',t)\phi_c'(\zeta'',t) + \frac{\xi(t)}{M_0+\xi(t)}\Pi_0'(x,t) \\
&+ \frac{M_0\phi_c'(\zeta)}{(M_0+\xi(t))^2}\int dx'\ \Pi_0'(x',t)\Phi'(x,t) - \frac{\phi_c'(\zeta')}{M_0+\xi(t)}\int dx'\Pi_0'(x',t)\phi_c'(\zeta')\ ,
\end{aligned} \tag{G.21}$$

where the second of these equations was obtained after a bit of algebra. In Eq. (G.20) we solve for $C(x',t',t)$ and substitute this into Eq. (G.21) which, after some manipulations, gives us

$$\begin{aligned}
0 &= \int dx'' D(x',t',x'',t)\left[\psi'(\zeta',t) - \frac{\xi(t)}{M_0}\phi_c'(\zeta'')\right] \\
&+ \frac{\phi_c'(\zeta)}{M_0+\xi(t)}\int dx''\Pi_0'(x'',t)\left[\psi'(\zeta',t) - \frac{\xi(t)}{M_0}\phi_c'(\zeta'')\right]\ ,
\end{aligned} \tag{G.22}$$

from which we deduce

$$D(x',t',x'',t'') = -\frac{\phi_c'(\zeta')}{M_0+\xi(t)}\Pi_0'(x'',t)\ . \tag{G.23}$$

Substitution of this expression for $D$ into Eq. (G.21) yields

$$C(x',t',t) = \Pi_0'(x',t)\delta(t-t') \tag{G.24}$$

Collecting these calculations we have

$$\begin{aligned}
\frac{\delta}{\delta\Phi(x',t')} &= -\frac{\phi_c'(\zeta')}{M_0+\xi(t')}\frac{\delta}{\delta X(t'')} + \int dx''\left\{\delta(x'-x'') - \frac{\phi_c'(\zeta')\psi'(\zeta'')}{M_0+\xi(t)}\right\}\frac{\delta}{\delta\psi(\zeta'')} \\
&+ \Pi_0'(x',t)\frac{\delta}{\delta p(t')} - \frac{\phi_c'(\zeta')}{M_0+\xi(t')}\int\ dx''\ \Pi_0'(x'',t)\frac{\delta}{\delta\pi(\zeta'')}\ .
\end{aligned} \tag{G.25}$$

We derive the analogous expression for the $\Pi_0$ derivative by using the same methods. For the sake of brevity we merely present the result,

$$\frac{\delta}{\delta\Pi_0(x',t')} = -\Phi'(x',t')\frac{\delta}{\delta p(t')} + \frac{\delta}{\delta\pi(\zeta')}\ , \tag{G.26}$$

where in both of the final expressions the full fields $\Phi$ and $\Pi_0$ are used to achieve a more compact notation.

# Appendix H

# Fokker-Planck Equation for $P(X, p; t)$

In this appendix we derive a Fokker Planck equation for the phase space distribution function $P(X, p; t)$ (see Chapter 6) by starting from the full field equation

$$\frac{\partial P(\Phi, \Pi_0; t)}{\partial t}$$
$$= \int_{-\infty}^{\infty} dx \Big\{ -\Pi_0 \frac{\delta}{\delta \Phi} P(\Phi, \Pi_0; t) - \frac{\delta}{\delta \Pi_0} \Big[ \Big( \Phi_{xx} - U'[\Phi] - \epsilon \Pi_0 \Big) P(\Phi, \Pi_0; t) \Big]$$
$$+ \quad \epsilon k_B T \frac{\delta^2}{\delta \Pi_0^2} P(\Phi, \Pi_0; t) \Big\} , \tag{H.1}$$

substituting the ansatz

$$P[\Phi, \Pi_0; t] = e^{-\beta H_{ph}} P(X, p; t) , \tag{H.2}$$

with

$$H_{ph} = \int \Big[ \frac{1}{2}\pi^2 + \frac{1}{2}\psi'^2 + \frac{1}{2}\psi^2 U''(\phi_c) \Big] . \tag{H.3}$$

and then using the results of Appendix G, we take the functional derivatives in Eq. (H.1) in terms of the kink variables. We shall consider each term in Eq. (H.1) separately to avoid extremely long expresssions. Using Eq. (G.25) the first term becomes

$$-\int dx \Pi_0 \frac{\delta P[\Phi, \Pi_0; t]}{\delta \Phi} =$$
$$- \quad \int dx \; \Pi_0 \Big\{ -\frac{\phi_c'(\zeta)}{M_0 + \xi} \frac{\delta P(X, p; t)}{\delta X} - \beta \int dx'' \Big[ \delta(x - x'') - \frac{\phi_c'(\zeta)\psi'(\zeta'')}{M_0 + \xi} \Big] \times$$
$$\times \quad P(X, p; t) \frac{\delta H_{ph}}{\delta \psi(\zeta'', t)} + \beta \frac{\phi_c'(\zeta)}{M_0 + \xi} \int dx'' \; \Pi_0' \frac{\delta H_{ph}}{\delta \pi(\zeta'', t)} \Big\} . \tag{H.4}$$

First consider the derivative with respect to $\psi$

$$
\begin{aligned}
& \frac{\delta H_{ph}}{\delta\psi(\zeta'',t)} \\
&= \int dx' \frac{\delta}{\delta\psi(\zeta'',t)} \Big[\frac{1}{2}\psi'^2(\zeta',t) + \frac{1}{2}\psi^2(\zeta',t)U''(\phi_c)\Big] && \text{(H.5)} \\
&= \int dx' \Big[-\psi''(\zeta',t) + \psi(\zeta',t)U''(\phi_c)\Big]\Big[\delta(x'-x'') - \frac{\phi_c'(\zeta')\phi_c'(\zeta'')}{M_0}\Big] && \text{(H.6)} \\
&= -\psi''(\zeta'',t) + \psi(\zeta'',t)U''[\phi_c(\zeta'')] \ , && \text{(H.7)}
\end{aligned}
$$

where we have repeatedly made use of the constraints (Eq. (3.3.10-11)) and the fact that $\phi_c'' = U'(\phi_c)$. Substituting this expression into the first integration over $x''$ in Eq. (H.4), we have

$$
\begin{aligned}
& \int dx'' \Big[\delta(x-x'') - \frac{\phi_c'(\zeta)\psi'(\zeta'')}{M_0+\xi}\Big] \frac{\delta H_{ph}}{\delta\psi(\zeta'',t)} = \\
& - \ \psi''(\zeta,t) + \psi(\zeta,t)U''[\phi_c(\zeta)] \\
& - \ \phi_c'(\zeta)\frac{\xi}{M_0}\Big[-\int \phi_c'\psi'' + \phi_c'\psi U''(\phi_c)\Big] && \text{(H.8)} \\
& = \ -\psi''(\zeta,t) + \psi(\zeta,t)U''[\phi_c(\zeta)] \ . && \text{(H.9)}
\end{aligned}
$$

Next we examine the $\pi$ derivative term:

$$
-\int dx \Pi_0(x,t) \frac{\phi_c(\zeta')}{M_0+\xi} \int dx'' \Pi_0'(x'',t) \frac{\delta H_{ph}}{\delta\pi(\zeta'',t)} \ . \tag{H.10}
$$

The derivative of $H_{ph}$ with respect to $\pi$ will bring down another factor of $\pi$ which results in the product of three momentum fields. Keeping terms of this order is not consistent with the phonon ansatz made and therefore we do not include this term. Collecting these results we have for the first term

$$
\begin{aligned}
& \int_{-\infty}^{\infty} -\Pi_0 \frac{\delta}{\delta\Phi} P(\Phi,\Pi_0;t) \\
&= \frac{M_0(p+\int \pi\psi')}{(M_0+\xi)^2} e^{-\beta H_{ph}} \frac{\delta P(X,p;t)}{\delta X} + \beta e^{-\beta H_{ph}} P(X,p;t) \int \Pi_0(\psi'' - \psi U'(\phi_c)) \ . && \text{(H.11)}
\end{aligned}
$$

Using the fact that

$$
\frac{\delta H_{ph}}{\delta \Pi_0} = \pi \ , \tag{H.12}
$$

we easily find that

$$
\begin{aligned}
&-\int dx \frac{-\delta}{\delta\Pi_0}\Big[\Phi_{xx} - U'(\Phi) - \epsilon\Pi_0\Big] e^{-\beta H_{ph}} P(X,p;t) = \\
&- \quad \beta P(X,p;t) e^{-\beta H_{ph}} \int dx\ \pi\Big[-\psi'' + \psi U'(\phi_c)\Big] + \epsilon \int dx\ \frac{-\delta}{\delta\Pi_0}\Big[\pi e^{-\beta H_{ph}} P(X,p;t)\Big] \\
&+ \quad \epsilon\ e^{-\beta H_{ph}} \frac{\delta}{\delta p} p P(X,p;t)\ . \qquad \text{(H.13)}
\end{aligned}
$$

Finally we have for the last term

$$
\begin{aligned}
&\frac{\epsilon}{\beta} \int dx \frac{\delta^2}{\delta\Pi_0^2}\Big[e^{-\beta H_{ph}} P(X,p;t)\Big] \\
&= \quad -\epsilon \int dx \frac{\delta}{\delta\Pi_0}\Big[\pi e^{-\beta H_{ph}} P(X,p;t)\Big] - \frac{\epsilon}{\beta} \int dx \frac{\delta}{\delta\Pi_0}\Big[\Phi' e^{-\beta H_{ph}} \frac{\delta P(X,p;t)}{\delta p}\Big] \qquad \text{(H.14)} \\
&= \quad -\epsilon \int dx \frac{\delta}{\delta\Pi_0}\Big[\pi e^{-\beta H_{ph}} P(X,p;t)\Big] \\
&+ \quad \frac{\epsilon}{\beta}\Big[\int \Phi'\Phi'\Big] e^{-\beta H_{ph}} \frac{\delta^2 P(X,p;t)}{\delta p^2} + \epsilon\Big(\int \psi'\pi\Big) e^{-\beta H_{ph}} \frac{\delta P(X,p;t)}{\delta p}\ . \qquad \text{(H.15)}
\end{aligned}
$$

Combining all three contributions we have

$$
\begin{aligned}
&e^{-\beta H_{ph}} \frac{\partial P(X,p;t)}{\partial t} \\
&= \quad e^{-\beta H_{ph}} \Bigg\{ \frac{p + \int \pi\psi'}{M_0(1+\xi/M_0)^2} \frac{\delta P(X,p;t)}{\delta X} \\
&+ \quad \beta \frac{p + \int \pi\psi'}{M_0(1+\xi/M_0)} \int dx \phi_c'\Big(\psi'' - \psi U'(\Phi)\Big) - \frac{p + \int \pi\psi'}{(M_0+\xi)^2} P(X,p;t) \\
&+ \quad \epsilon \frac{\delta}{\delta p}\Big[p P(X,p;t)\Big] + \frac{\epsilon}{\beta}\Big(\int \Phi'\Phi'\Big) \frac{\delta^2 P(X,p;t)}{\delta p^2} \Bigg\}\ . \qquad \text{(H.16)}
\end{aligned}
$$

# Appendix I

# Potentials and Masses for Kink-Antikink $\phi^4$ Collisions

Since the analytic expressions for the potentials $V_i(x_0, y_0)$ and masses $m_i(x_0, y_0)$ in Chapter 7 are somewhat lengthy, we include them here, along with some useful properties and Taylor series. Since the integrals have already been published [15], we merely reproduce these analytic expressions, pointing out an error of 1/2 in the mass $m_2$. Since the "relativistic" calculations given involve integrals similar to those done by Campbell et al. [15], we recast the expressions in terms of three functions $w_i(\, z_0)$ with $z_0$ given by

$$z_0 \equiv \frac{m x_0 y_0}{\sqrt{2}\lambda} \; . \tag{I.1}$$

In terms of these functions $w_i$ the potentials and masses take the form:

$$V_1 = \frac{\sqrt{2} m^3 y_0}{2\lambda} \left[ \frac{4}{3} - w_1(z_0) \right] , \tag{I.2}$$

$$V_2 = \frac{\sqrt{2} m^3}{4\lambda y_0} w_2(z_0) \; , \tag{I.3}$$

$$m_1 = \frac{\sqrt{2} m^3 y_0}{\lambda} \left[ \frac{4}{3} + w_1(z_0) \right] , \tag{I.4}$$

$$m_2 = \frac{\sqrt{2} m^3 x_0}{\lambda} w_1(z_0) \; , \tag{I.5}$$

$$m_3 = \frac{\sqrt{2} m^3}{4\lambda y_0} w_2(z_0) \; , \tag{I.6}$$

with the functions $w_i$ given by

$$w_1(z_0) = \frac{\operatorname{sech}(z_0)(1 + \tanh^2(z_0))}{\tanh^3(z_0)} \Big( z_0 - \frac{\tanh(z_0)}{1 + \tanh^2(z_0)} \Big) \; , \tag{I.7}$$

$$\begin{aligned}
w_2(z_0) &= \frac{16(1+\tanh^2(z_0))}{\tanh(z_0)}\Big(z_0 - \frac{\tanh(z_0)}{1+\tanh^2(z_0)}\Big) \\
&- \frac{8(1+\tanh^2(z_0))^2}{\tanh^2(z_0)}\Bigg[\Big(3 - \frac{4\tanh^2(z_0)}{1+\tanh^2(z_0)^2}\Big)z_0 - \frac{3\tanh(z_0)}{1+\tanh^2(z_0)}\Bigg] \\
&+ \frac{2(1+\tanh^2(z_0))^3}{\tanh^3(z_0)}\Bigg[\Big(5 - \frac{12\tanh^2(z_0)}{1+\tanh^2(z_0)^2}\Big)z_0 + \frac{16\tanh^3(z_0)}{3(1+\tanh^2(z_0)^3)} \\
&\qquad - \frac{5\tanh(z_0)}{1+\tanh^2(z_0)}\Bigg] , \qquad \text{(I.8)} \\
w_3(z_0) &= \frac{1}{3}\left(\frac{\pi^2}{6} - 1\right) \\
&- \frac{2}{\sinh^2(2z_0)}\left[\frac{\pi^2}{12}\Big(\frac{2z_0}{\tanh(2z_0)} - 1\Big) - \frac{4z_0^3}{3\tanh(2z_0)}\right] . \qquad \text{(I.9)}
\end{aligned}$$

With the aide of MACSYMA [136] the following Taylor series were computed:

$$\begin{aligned}
w_1(z_0) &\approx \frac{4}{3} - \frac{32}{15}z_0^2 + \frac{128}{63}\,z_0^4 - \frac{1024}{675}\,z_0^6 \\
&+ \frac{2048}{2079}\,z_0^8 - \frac{11321344}{19348875}\,z_0^{10} + \frac{65536}{200475}\,z_0^{12} , \qquad \text{(I.10)} \\
w_2(\,z_0) &\approx \frac{64}{3}\,z_0^2 - \frac{512}{15}\,z_0^3 + \frac{2816}{315}\,z_0^4 + \frac{2048}{189}\,z_0^5 \\
&- \frac{2048}{189}\,z_0^6 - \frac{16384}{1575}\,z_0^7 - \frac{421888}{51975}\,z_0^8 + \frac{32768}{6237}\,z_0^9 \\
&+ \frac{219250688}{42567525}\,z_0^{10} - \frac{181141504}{70945875}\,z_0^{11} + \frac{1906180096}{638512875}\,z_0^{12} , \qquad \text{(I.11)} \\
w_3(\,z_0) &\approx \frac{4\pi^2}{45}\,z_0^2 + \frac{80\pi^2+336}{945}\,z_0^4 + \frac{896\pi^2+6400}{14175}z_0^6 - \frac{6400\pi^2+59136}{675}\,z_0^8 \\
&+ \frac{15566848\pi^2+167731200}{638512875}\,z_0^{10} - \frac{5218304\pi^2+62267392}{383107725}\,z_0^{12} . \qquad \text{(I.12)}
\end{aligned}$$

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$$\frac{1}{\pi}\int_0^\infty \frac{dt}{t}\sin[at+\frac{b}{t}] = J_0(2\sqrt{ab})$$

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